

THE WEYL-TYPE ASYMPTOTIC FORMULA FOR BIHARMONIC STEKLOV EIGENVALUES WITH DIRICHLET BOUNDARY CONDITION ON RIEMANNIAN MANIFOLDS

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ABSTRACT. Let Ω be a bounded domain with C^2 -smooth boundary in an n -dimensional oriented Riemannian manifold. It is well-known that for the bi-harmonic equation $\Delta^2 u = 0$ in Ω with the 0-Dirichlet boundary condition, there exists an infinite set $\{u_k\}$ of biharmonic functions in Ω with positive eigenvalues $\{\lambda_k\}$ satisfying $\Delta u_k + \lambda_k \varrho \frac{\partial u_k}{\partial \nu} = 0$ on the boundary $\partial\Omega$. In this paper, by a new method we establish the Weyl-type asymptotic formula for the counting function of the biharmonic Steklov eigenvalues λ_k .

1. INTRODUCTION

Spectral asymptotics for partial differential operators have been the subject of extensive research for over a century. It has attracted the attention of many outstanding mathematicians and physicists. Beyond the beautiful asymptotic formulas that are intimately related to the geometric properties of the domain and its boundary, a sustaining force has been its important role in mathematics, mechanics and theoretical physics (see, for example, [4], [5], [6], [7], [9], [18], [19], [20], [28], [32], [33], [36], [39], [40], [41], [42], [50]).

Let (\mathcal{M}, g) be an oriented Riemannian manifold of dimension n with a positive definite metric tensor g , and let $\Omega \subset \mathcal{M}$ be a bounded domain with C^2 -smooth boundary $\partial\Omega$. Assume ϱ is a non-negative bounded function defined on $\partial\Omega$. We consider the following classical biharmonic Steklov eigenvalue problem:

$$(1.1) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta_g u + \lambda \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν denotes the inward unit normal vector to $\partial\Omega$, and Δ_g is the Laplace-Beltrami operator defined in local coordinates by the expression,

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right).$$

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Here $|g| := \det(g_{ij})$ is the determinant of the metric tensor, and g^{ij} are the components of the inverse of the metric tensor g .

The problem (1.1) has nontrivial solutions u only for a discrete set of $\lambda = \lambda_k$, which are called biharmonic Steklov eigenvalues (see [11], [21], [32] or [47]). Let us enumerate the eigenvalues in increasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots,$$

where each eigenvalue is counted as many times as its multiplicity. The corresponding eigenfunctions $\frac{\partial u_1}{\partial \nu}, \frac{\partial u_2}{\partial \nu}, \dots, \frac{\partial u_k}{\partial \nu}, \dots$ form a complete orthonormal basis in $L^2_\varrho(\partial\Omega)$ (see, Proposition 3.5). It is clear that λ_k can be characterized variationally as

$$\lambda_1 = \frac{\int_\Omega |\Delta_g u_1|^2 dR}{\int_{\partial\Omega} \varrho \left(\frac{\partial u_1}{\partial \nu} \right)^2 ds} = \inf_{\substack{v \in H_0^1(\Omega) \cap H^2(\Omega) \\ 0 \neq \frac{\partial v}{\partial \nu} \in L^2(\partial\Omega)}} \frac{\int_\Omega |\Delta_g v|^2 dR}{\int_{\partial\Omega} \varrho \left(\frac{\partial v}{\partial \nu} \right)^2 ds},$$

$$\lambda_k = \frac{\int_\Omega |\Delta_g u_k|^2 dR}{\int_{\partial\Omega} \varrho \left(\frac{\partial u_k}{\partial \nu} \right)^2 ds} = \max_{\substack{\mathcal{F} \subset H_0^1(\Omega) \cap H^2(\Omega) \\ \text{codim}(\mathcal{F})=k-1}} \inf_{\substack{v \in \mathcal{F} \\ 0 \neq \frac{\partial v}{\partial \nu} \in L^2(\partial\Omega)}} \frac{\int_\Omega |\Delta_g v|^2 dR}{\int_{\partial\Omega} \varrho \left(\frac{\partial v}{\partial \nu} \right)^2 ds}, \quad k = 2, 3, 4, \dots$$

where $H^m(\Omega)$ is the Sobolev space, and where dR and ds are the Riemannian elements of volume and area on Ω and $\partial\Omega$, respectively.

In elastic mechanics, when the weight of the body Ω is the only body force, the stress function u must satisfy the equation $\Delta^2 u = 0$ in Ω (see, p.32 of [45]). In addition, the boundary condition in (1.1) has an interesting interpretation in theory of elasticity. Consider the model problem (see [11]):

$$(1.2) \quad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0, \quad \Delta u + (1 - \sigma)\iota \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary, $\sigma \in (-1, 1/2)$ is the Poisson ratio and ι is the mean curvature of the boundary $\partial\Omega$. Problem (1.2) describes the deformation u of the linear elastic supported plate Ω under the action of the transversal exterior force $f = f(x)$, $x \in \Omega$. The Poisson ratio σ of an elastic material is the negative transverse strain divided by the axial strain in the direction of the stretching force. In other words, this parameter measures the transverse expansion (respectively, contraction) if $\sigma > 0$ (respectively, $\sigma < 0$) when the material is compressed by an external force. We refer to [22], [46] for more details. The restriction on the Poisson ratio is due to thermodynamic considerations of strain energy in the theory of elasticity. As shown in [22], there exist materials for which the Poisson ratio is negative and the limit case $\sigma = -1$ corresponds to materials with an infinite flexural rigidity (see, p.456 of [41]). This limit value for σ is strictly related to the eigenvalue problem (1.1). Hence, the limit value $\sigma = -1$, which is not allowed from a physical point of view, also changes the structure of the stationary problem (1.2): For example (see [11]), when Ω is the unit disk and $\lambda_1 = (1 - \sigma)\iota = 1 - \sigma = 2$, (1.2) either admits an infinite number of solutions or it admits no solutions at all, depending on f .

Problem (1.1) is also important in conductivity and biharmonic analysis because the related problem was initially studied by Calderón (cf. [3]). This connection arises because the set of the eigenvalues for the biharmonic Steklov problem is the same as the set of eigenvalues of the well-known “Neumann-to-Laplacian” map for biharmonic equation (This map associates each normal derivative $\partial u / \partial \nu$ defined on the boundary $\partial\Omega$ to the restriction $(\Delta u)|_{\partial\Omega}$ of the Laplacian of u for the biharmonic function u on Ω , where the biharmonic function u is uniquely determined by $u|_{\partial\Omega} = 0$ and $(\partial u / \partial \nu)|_{\partial\Omega}$).

In the general case the eigenvalues λ_k can not be evaluated explicitly. In particular, for large k it is difficult to calculate them numerically. In view of the important applications, one is interested in finding the asymptotic formulas for λ_k as $k \rightarrow \infty$. However, for a number of reasons it is traditional in such problems to deal with the matter the other way round, i.e., to study the sequential number k as a function of τ . Namely, let us introduce the counting function $A(\tau)$ defined as the number of eigenvalues λ_k less than or equal to a given τ . Then our asymptotic problem is reformulated as the study of the asymptotic behavior of $A(\tau)$ as $\tau \rightarrow +\infty$.

In order to better understand our problem (1.1) and its asymptotic behavior, let us mention the Steklov eigenvalue problem for the harmonic equation

$$(1.3) \quad \begin{cases} \Delta_g v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} + \eta \varrho v = 0 & \text{on } \partial\Omega, \end{cases}$$

where η is a real number. This problem was first introduced by V. A. Steklov for bounded domains in the plane in [41] (The reader should be aware that “Steklov” is also often transliterated as “Stekloff”.) His motivation came from physics. The function v represents the steady state temperature on Ω such that the flux on the boundary is proportional to the temperature (In two dimensions, it can also be interpreted as a membrane with whole mass concentrated on the boundary). For the harmonic Steklov eigenvalue problem (1.3), in a special case in two dimensions, Å. Pleijel [35] outlined an investigation of the asymptotic behavior of both eigenvalues and the eigenfunctions. In 1955, L. Sandgren [39] established the asymptotic formula of the counting function $B(\tau) = \#\{\eta_k | \eta_k \leq \tau\}$:

$$(1.4) \quad B(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(2\pi)^{n-1}} \int_{\partial\Omega} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty,$$

i.e.,

$$\lim_{\tau \rightarrow +\infty} \frac{B(\tau)}{\tau^{n-1}} = \frac{\omega_{n-1}}{(2\pi)^{n-1}} \int_{\partial\Omega} \varrho^{n-1} ds,$$

where ω_{n-1} is the volume of the unit ball of \mathbb{R}^{n-1} , and the integral is over the boundary $\partial\Omega$. This asymptotic behavior is motivated by the similar one for the eigenvalues of the Dirichlet Laplacian. The classical result for the Dirichlet (or Neumann) eigenvalues of the Laplacian on a smooth bounded domain is Weyl’s formula (see [4], [6] or [49]):

$$(1.5) \quad N(\tau, \Omega) \sim \frac{\omega_n}{(2\pi)^n} (\text{vol}(\Omega)) \tau^{n/2} \quad \text{as } \tau \rightarrow +\infty,$$

where $N(\tau, \Omega)$ is the number of the Dirichlet (or Neumann) eigenvalues of domain Ω less than or equal to a given τ . In the case of two-dimensional Euclidean space, Pleijel [34] in 1950 proved an asymptotic formula for the eigenvalues Ξ_k^2 of a clamped plate problem:

$$(1.6) \quad \begin{cases} \Delta^2 u - \Xi^2 u = 0 & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Grub[15] and Ashbaugh, Gesztesy, Mitrea and Teschl [2] obtained Weyl’s asymptotic formula for the eigenvalues Λ_k of the buckling problem in \mathbb{R}^n :

$$(1.7) \quad \begin{cases} \Delta^2 u + \Lambda \Delta u = 0 & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that for the Dirichlet eigenvalues, the Neumann eigenvalues, the buckling eigenvalues and the square root of the clamped plate eigenvalues in a fixed domain, their counting functions have the same asymptotic formula (1.5) (see, for example, [4], [25], [15], [48] and [49]).

The study of asymptotic behavior of the biharmonic Steklov eigenvalues is much more difficult than that of the harmonic Steklov eigenvalues. It had been a challenging problem in the past 50 years. The main stumbling block that lies in the way is the estimates for the distribution of the boundary eigenvalues for bi-harmonic equations with suitable boundary conditions. Some important works have contributed to the research of this problem, for example, L. E. Payne [32], J. R. Kuttler and V. G. Sigillito [21], A. Ferrero, F. Gazzola and T. Weth [11], Q. Wang and C. Xia [47], and others.

In this paper, by a new method we establish the Weyl-type asymptotic formula for the counting function of the biharmonic Steklov eigenvalues. The main result is the following:

Theorem 1.1. *Let (\mathcal{M}, g) be an n -dimensional oriented Riemannian manifold, and let $\Omega \subset \mathcal{M}$ be a bounded domain with C^2 -smooth boundary $\partial\Omega$. Then*

$$(1.8) \quad A(\tau) \sim \frac{\omega_{n-1}\tau^{n-1}}{(4\pi)^{(n-1)}} \int_{\partial\Omega} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty,$$

where $A(\tau)$ is defined as before.

Corollary 1.2. *Let (\mathcal{M}, g) be an n -dimensional oriented Riemannian manifold, and let $\Omega \subset \mathcal{M}$ be a bounded domain with C^2 -smooth boundary $\partial\Omega$. If, in problem (1.1), $\varrho \equiv 1$ on $\partial\Omega$, then*

$$(1.9) \quad \lambda_k \sim (4\pi) \left(\frac{k}{\omega_{n-1}(\text{vol}(\partial\Omega))} \right)^{1/(n-1)} \quad \text{as } k \rightarrow +\infty.$$

We outline the idea of the proof of Theorem 1.1. First, we make a division of $\bar{\Omega}$ into subdomains (by dividing $\partial\Omega$ into sufficiently small parts, then taking a depth $\sigma > 0$ (small enough) in the direction of inner normal of $\partial\Omega$ to form a finite number of n -dimensional subdomains). Under a sufficiently fine division of $\partial\Omega$ (also σ sufficiently small), g^{ik} and ϱ can be replaced by constants because their variant will be small, so that the corresponding subdomains whose partial boundaries are situated at the $\partial\Omega$ can be approximated by Euclidean cylinders. Next, we construct three Hilbert spaces of functions and their self-adjoint linear transformations whose eigenfunctions are just the Steklov eigenfunctions with corresponding boundary conditions. It can be shown that these Steklov eigenvalue problems have the same boundary conditions on the base of each cylinder as the original one in problem (1.1) but they have relevant boundary conditions on the other parts of a cylinder. In particular, on each cylindrical surface, these boundary conditions will be one of the three forms $u = \Delta_g u = 0$, $\Delta_g u = \frac{\partial(\Delta_g u)}{\partial\nu} = 0$ and $\frac{\partial u}{\partial\nu} = \frac{\partial(\Delta_g u)}{\partial\nu} = 0$. The main purpose of constructing such Steklov problems is that when putting together such cylinders, we can obtain global upper and lower estimates for the counting function $A(\tau)$ of the original Steklov problem (i.e., $A^0(\tau) \leq A(\tau) \leq A^d(\tau) \leq A^f(\tau)$ for all $\tau > 0$, see Sections 3, 6). For each Euclidean cylinder, by using a cubical net we can divide the base of the cylinder into $(n-1)$ -dimensional cubes and some smaller parts which intersect boundary of the base, so that we get n -dimensional parallelepipeds and some smaller n -dimensional cylinders. As for the n -dimensional parallelepiped, we can explicitly calculate the Steklov eigenfunctions and eigenvalues by separating variables, and then we can compute the asymptotic distribution of eigenvalues by means of the well-known variational methods used by H. Weyl [50], R. Courant and D. Hilbert [6] in the case of the membranes. Meanwhile, for each small n -dimensional cylinder, by introducing a nice transformation we may map it into a special cylinder whose counting functions of Steklov eigenvalues can also be estimated. Finally, applying normal coordinates system at a fixed point of each subdomain of a division and

combining these estimates, we establish the desired asymptotic formula for $A(\tau)$. Note that the Holmgren uniqueness theorem for the solutions of elliptic equations plays a crucial role in this paper.

This paper is organized as follows. In Section 2, we prove two compact trace lemmas for bounded domains with piecewise smooth boundaries. In Section 3, we define various self-adjoint transformations on the associated Hilbert spaces of functions, and give the connections between the eigenfunctions of self-adjoint transformations and the Steklov eigenfunctions (corresponding to different kinds of boundary conditions). Section 4 is dedicated to deriving the explicit formulas for the biharmonic Steklov eigenvalues and eigenfunctions in an n -dimensional rectangular parallelepiped of \mathbb{R}^n , which depends on a key calculation for the solutions of biharmonic equations. The counting functions of Steklov eigenvalues for general cylinder of the Euclidean space are dealt with in Section 5. In the final section, we prove Theorem 1.1 and Corollary 1.2 on Riemannian manifolds.

2. COMPACT TRACE LEMMAS

An n -dimensional cube in \mathbb{R}^n is the set $\{x \in \mathbb{R}^n \mid 0 \leq x_i \leq a, i = 1, \dots, n\}$.

Let f be a real-valued function defined in an open set Ω in \mathbb{R}^n ($n \geq 1$). For $y \in \Omega$ we call f *real analytic at y* if there exist $a_\beta \in \mathbb{R}^1$ and a neighborhood U of y (all depending on y) such that

$$f(x) = \sum_{\beta} a_{\beta} (x - y)^{\beta}$$

for all x in U . We say f is *real analytic in Ω* , if f is real analytic at each $y \in \Omega$.

Let Ω together with its boundary be transformed pointwise into the domain Ω' together with its boundary by equations of the form

$$(2.1) \quad x'_i = x_i + f_i(x_1, \dots, x_n), \quad i = 1, 2, \dots, n.$$

where the functions f_i and their first order derivatives are Lipschitz continuous throughout the domain, and they are less in absolute value than a small positive number ϵ . Then we say that the domain Ω is approximated by the domain Ω' with the degree of accuracy ϵ .

Let (\mathcal{M}, g) be a Riemannian manifold. A subset Γ of (\mathcal{M}, g) is said to be an $(n-1)$ -dimensional smooth (respectively, real analytic) surface if Γ is nonempty and if for every point x in Γ , there is a smooth (respectively, real analytic) diffeomorphism of the open unit ball $B(0, 1)$ in \mathbb{R}^n onto an open neighborhood U of x such that $B(0, 1) \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ maps onto $U \cap \Gamma$.

An $(n-1)$ -dimensional surface Γ in (\mathcal{M}, g) is said to be piecewise smooth (respectively, piecewise real analytic) if there exist a finite number of $(n-2)$ -dimensional smooth surfaces, by which Γ can be divided into a finite number of $(n-1)$ -dimensional smooth (respectively, real analytic) surfaces.

A subset \mathfrak{F} of $L^2(\Gamma)$ is called precompact if any infinite sequence $\{u_k\}$ of elements of \mathfrak{F} contains a Cauchy subsequence $\{u_{k'}\}$, i.e., one for which

$$(2.2) \quad \int_{\Gamma} (u_{k'} - u_{l'})^2 ds \rightarrow 0 \quad \text{as } k', l' \rightarrow \infty.$$

From here up to Section 5, let \mathcal{M} be an n -dimensional Riemannian manifold with real analytic metric tensor g .

Lemma 2.1. *Let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary. Assume that \mathfrak{M} is a set of functions u in $H_0^1(D) \cap H^2(D)$ for which*

$$(2.3) \quad \int_D |\Delta_g u|^2 dR$$

is uniformly bounded. Then the set $\{\frac{\partial u}{\partial \nu} | u \in \mathfrak{M}\}$ is precompact in $L^2(\partial D)$.

Proof. Put

$$(2.4) \quad \Lambda_1(D) = \inf_{u \in H_0^1(D) \cap H^2(D)} \frac{\int_D |\Delta_g u|^2 dR}{\int_D |\nabla_g u|^2 dR}.$$

We claim that $\Lambda_1(D) > 0$. In fact, by applying Green's formula (see, for example, [4] or [39]) and Schwarz's inequality we see that for any $u \in H_0^1(D) \cap H^2(D)$,

$$\left(\int_D |\nabla_g u|^2 dR \right)^2 = \left| \int_D -u(\Delta_g u) dR \right|^2 \leq \left(\int_D u^2 dR \right) \left(\int_D |\Delta_g u|^2 dR \right),$$

i.e.,

$$(2.5) \quad \frac{\int_D |\nabla_g u|^2 dR}{\int_D |u|^2 dR} \leq \frac{\int_D |\Delta_g u|^2 dR}{\int_D |\nabla_g u|^2 dR},$$

where

$$\int_D |\nabla_g u|^2 dR = \int_D g^{ik}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \sqrt{|g|} dx.$$

Since the first Dirichlet eigenvalue $\lambda_1(D)$ is positive for the bounded domain D , i.e.,

$$(2.6) \quad 0 < \lambda_1(D) = \inf_{u \in H_0^1(D)} \frac{\int_D |\nabla_g u|^2 dR}{\int_D |u|^2 dR},$$

we find by (2.5) and (2.6) that $\Lambda_1(D) > 0$, and the claim is proved.

From (2.6) and (2.4) we obtain that

$$(2.7) \quad \int_D |u|^2 dR \leq \frac{1}{\lambda_1(D)} \int_D |\nabla_g u|^2 dR \quad \text{for all } u \in H_0^1(D)$$

and

$$(2.8) \quad \int_D |\nabla_g u|^2 dR \leq \frac{1}{\Lambda_1(D)} \int_D |\Delta_g u|^2 dR \quad \text{for all } u \in H_0^1(D) \cap H^2(D).$$

Since ∂D is piecewise smooth, we can write $\partial D = \cup_{i=1}^m \Gamma_i$, where Γ_i is an $(n-1)$ -dimensional surface. For each fixed i , ($i = 1, \dots, m$), we choose a smooth $(n-1)$ -dimensional surface $\Gamma' \subset \subset D$ such that $\partial \Gamma'_i = \partial \Gamma_i$ and $\Gamma_i \cup \Gamma'_i$ bounds an n -dimensional Lipschitz domain D'_i satisfying $D'_i \subset \subset D \cup \Gamma_i$. Note that $u = 0$ on Γ_i for $u \in H_0^1(D) \cap H^2(D)$ (see, for example, p. 62 of [24] or Corollary 6.2.43 of [16]). It follows from the *a priori* estimate of the elliptic operators (see, for example, Theorem 9.13 of [13]) that there exists a constant $C_i > 0$ depending only on n, Γ_i, D'_i and D such that

$$(2.9) \quad \|u\|_{H^2(D'_i)} \leq C_i (\|\Delta u\|_{L^2(D)} + \|u\|_{L^2(D)}).$$

By assumption, we have $\int_D |\Delta u|^2 dR \leq \tilde{C}$ for all $u \in \mathfrak{M}$, where $\tilde{C} > 0$ is a constant. According to (2.7), (2.8) and (2.9), we see that for every $u \in \mathfrak{M}$,

$$(2.10) \quad \|u\|_{H^2(D'_i)} \leq C''_i,$$

where $C_i''' > 0$ is a constant depending only on n, Γ_i, D_i', D and \tilde{C} . Since D_i' is a domain with Lipschitz boundary in (\mathcal{M}, g) , it follows from the Neumann trace theorem (see, p. 16 of [2], p. 127 of [29], [14] or Chs V, VI of [8]) that

$$\left. \frac{\partial}{\partial \nu} \right|_{\Gamma_i} = \nu \cdot \nabla_g : \mathfrak{M} \rightarrow L^2(\Gamma_i)$$

is precompact for each i ($i = 1, \dots, m$). Consequently, we obtain that $\{\frac{\partial u}{\partial \nu}|u \in \mathfrak{M}\}$ is precompact in $L^2(\partial D)$. \square

Lemma 2.2. *Let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary. Suppose Γ_1 is a domain in ∂D with $\partial D - \Gamma_1 \neq \emptyset$ and assume that Γ_2 is an $(n-1)$ -dimensional real analytic surface in ∂D satisfying $\Gamma_2 \subset \subset \partial D - \Gamma_1$. Assume \mathfrak{E} is a set of functions u in $K^d(D) = \{u|u \in H^2(D), u = 0 \text{ on } \Gamma_1, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2\}$ for which*

$$(2.11) \quad \int_D |\Delta_g u|^2 dR$$

is uniformly bounded. Then the set $\{\frac{\partial u}{\partial \nu}|_{\Gamma_1} : u \in \mathfrak{E}\}$ is precompact in $L^2(\Gamma_1)$.

Proof. Since ∂D is piecewise smooth, it follows that Γ_1 can be divided into a finite number of smooth $(n-1)$ dimensional surfaces. Without loss of generality, we let Γ_1 itself be a smooth $(n-1)$ dimensional surface. Put

$$(2.12) \quad \lambda_{\Gamma_1}(D) = \inf_{v \in K^d(D), \int_D |v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |v|^2 dR}.$$

In order to prove the existence of a minimizer to (2.12), consider a minimizing sequence v_m in $K^d(D)$, i.e.,

$$\int_D |\Delta_g v_m|^2 dR \rightarrow \lambda_{\Gamma_1}(D) = 0 \quad \text{as } m \rightarrow +\infty$$

with $\int_D |v_m|^2 dR = 1$. Then, there is a constance $C > 0$ such that

$$(2.13) \quad \|\Delta_g v_m\|_{L^2(D)} \leq C, \quad \|v_m\|_{L^2(D)} \leq C \quad \text{for all } m.$$

Let $\{D_l\}$ be a sequence of Lipschitz domains such that $D_1 \subset D_2 \subset \dots \subset D_l \subset \dots \subset \subset D \cup \Gamma_1 \cup \Gamma_2$, $\cup_{l=1}^{\infty} D_l = D$, and $\Gamma_1 \cup \Gamma_2 \subset \partial D_l$ for all l . It follows from the *a priori* estimate for elliptic equations (see, for example, Theorem 9.13 of [13]) that there exists a constant $C_l' > 0$ depending only on n, D_l, D, Γ_1 and Γ_2 , such that

$$(2.14) \quad \|v_m\|_{H^2(D_l)} \leq C_l' (\|\Delta_g v_m\|_{L^2(D)} + \|v_m\|_{L^2(D)}).$$

From this and (2.13), we see that

$$\|v_m\|_{H^2(D_l)} \leq C_l'' \quad \text{for all } m,$$

where C_l'' is a constant depending only on $n, D_l, D, \Gamma_1, \Gamma_2$ and C . For each l , by the Banach-Alaoglu theorem we can then extract a subsequence $\{v_{l,m}\}_{m=1}^{\infty}$ of $\{v_m\}$, which converges weakly in $H^2(D_l)$ to a limit u , and converges strongly in $L^2(D_l)$ to u . We may assume that $\{v_{l+1,m}\}_{m=1}^{\infty}$ is a subsequence of $\{v_{l,m}\}_{m=1}^{\infty}$ for every l . Then, the diagonal sequence $\{v_{l,l}\}_{l=1}^{\infty}$ converges weakly in H^2 to u , and strongly converges to u in L^2 , in every compact

subset E of D . It is obvious that $\|u\|_{L^2(D)} = 1$. Since the functional $\int_{D_l} |\Delta_g u|^2 dR$ is lower semicontinuous in the weak $H^2(D_l)$ topology, we have

$$\int_{D_l} |\Delta_g u|^2 dR \leq \liminf_{k \rightarrow \infty} \int_{D_l} |\Delta_g v_{k,k}|^2 dR,$$

so that

$$\begin{aligned} \int_D |\Delta_g u|^2 dR &= \lim_{l \rightarrow \infty} \int_{D_l} |\Delta_g u|^2 dR \leq \lim_{l \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} \int_{D_l} |\Delta_g v_{k,k}|^2 dR \right) \\ &\leq \lim_{l \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} \int_D |\Delta_g v_{k,k}|^2 dR \right) = \lambda_{\Gamma_1}(D). \end{aligned}$$

For each fixed l , since $v_{k,k} \rightarrow u$ weakly in $H^2(D_l)$, we get that $v_{k,k} \rightarrow u$ strongly in $H^r(D_l)$ for any $0 < r < 2$. Note that $\frac{\partial v_{k,k}}{\partial \nu}|_{\Gamma_2} = 0$ and $v_{k,k}|_{\Gamma_i} = 0$ for $i = 1, 2$. It follows that $u|_{\Gamma_1} = 0$ and $u|_{\Gamma_2} = \frac{\partial u}{\partial \nu}|_{\Gamma_2} = 0$. Therefore $u \in K^d(D)$ is a minimizer.

We claim that $\lambda_{\Gamma_1}(D) > 0$. Suppose by contradiction that $\lambda_{\Gamma_1}(D) = \frac{\int_D |\Delta_g u|^2 dR}{\int_D |u|^2 dR} = 0$. Then $\Delta_g u = 0$ in D . Since the coefficients of the Laplacian are real analytic in D , and since Γ_2 is a real analytic surface, we find with the aid of the regularity for elliptic equations (see, Theorem A of [31], [30] or [1]) that u is real analytic up to the partial boundary Γ_2 . Note that $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_2 . Applying Holmgren's uniqueness theorem (see, Corollary 5 of p. 39 in [37] or p. 433 of [43]) for the real analytic elliptic equation $\Delta_g u = 0$ in D , we get $u \equiv 0$ in D . This contradicts the fact $\int_D |u|^2 dR = 1$, and the claim is proved. Therefore we have

$$(2.15) \quad \int_D |u|^2 dR \leq \frac{1}{\lambda_{\Gamma_1}(D)} \int_D |\Delta_g u|^2 dR \quad \text{for } u \in K^d(D).$$

According to the assumption, there is a constant C'' such that

$$(2.16) \quad \|\Delta_g u\|_{L^2(D)} \leq C'' \quad \text{for all } u \in \mathfrak{E}.$$

Again, applying the *a priori* estimate for the elliptic equations in some (fixed) subdomain $D_l \subset \subset D \cup \Gamma_1 \cup \Gamma_2$ (see, Theorem 9.13 of [13]), we obtain that

$$(2.17) \quad \|u\|_{H^2(D_l)} \leq C'_l (\|\Delta_g u\|_{L^2(D)} + \|u\|_{L^2(D)}),$$

where the constant C'_l is as in (2.14). By (2.15)–(2.17), we get that for every $u \in \mathfrak{E}$,

$$\|u\|_{H^2(D_l)} \leq C''',$$

where $C''' > 0$ is a constant depending only on $n, D_l, D, \Gamma_1, \Gamma_2$ and C'' . It follows from the Neumann trace theorem (see, p. 16 of [2], [14] or [27]) that $\{\frac{\partial u}{\partial \nu}|_{\Gamma_1} : u \in \mathfrak{E}\}$ is precompact in $L^2(\Gamma_1)$. \square

The following two results will be needed later:

Proposition 2.3 (see, p. 12 of [39]). *Let Π^0 be an isometric transformation which maps a Hilbert space \mathcal{H}^0 onto a subspace $\Pi^0 \mathcal{H}^0$ of another Hilbert space \mathcal{H} , so that*

$$\langle u^0, v^0 \rangle^0 = \langle \Pi^0 u^0, \Pi^0 v^0 \rangle \quad \text{for all } u^0, v^0 \in \mathcal{H}^0.$$

Suppose that G^0 and G are two non-negative, self-adjoint, completely continuous linear transformations on \mathcal{H}^0 and \mathcal{H} respectively, such that

$$\langle G^0 u^0, v^0 \rangle^0 = \langle G \Pi^0 u^0, \Pi^0 v^0 \rangle \quad \text{for all } u^0, v^0 \in \mathcal{H}^0.$$

Then

$$\mu_k^0 \leq \mu_k \quad \text{for } k = 1, 2, 3, \dots,$$

where $\{\mu_k^0\}$ and $\{\mu_k\}$ are the eigenvalues of G^0 and G , respectively.

Proposition 2.4 (see, p. 13 of [39]). *Assume that \mathcal{H} is a direct sum of p Hilbert spaces \mathcal{H}_j*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_p$$

and that the self-adjoint, completely continuous linear transformation G maps every \mathcal{H}_j into itself,

$$G\mathcal{H}_j \subset \mathcal{H}_j, \quad j = 1, 2, 3, \dots, p.$$

Denote by G_j the restriction of G to \mathcal{H}_j . Then the set of eigenvalues of the transformation G (each eigenvalue repeated according to its multiplicity) is identical to the union of the sets of eigenvalues of G_1, \dots, G_p .

3. COMPLETELY CONTINUOUS TRANSFORMATIONS AND EIGENVALUES

Let (\mathcal{M}, g) be an n -dimensional real analytic Riemannian manifold and let $D \subset \mathcal{M}$ be a bounded domain with piecewise smooth boundary Γ . Suppose that ϱ is a non-negative bounded function defined on Γ or only on a portion Γ_ϱ of Γ (measure $\Gamma_\varrho = \int_{\Gamma_\varrho} ds > 0$) and assume that $\int_{\Gamma_\varrho} \varrho ds > 0$. In case $\Gamma_\varrho \neq \Gamma$ we denote $\Gamma_0 = \Gamma - \bar{\Gamma}_\varrho$, and assume that Γ_{00} is a real analytic $(n-1)$ -dimensional surface in Γ_0 .

If $\Gamma_\varrho \neq \Gamma$ (measure $\Gamma_0 > 0$), we denote

$$\begin{aligned} K(D) &= \{u \mid u \in H_0^1(D) \cap H^2(D), \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_{00}\} \\ K^d(D) &= \{u \mid u \in H^2(D), u = 0 \text{ on } \Gamma_\varrho, \text{ and } u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_{00}\}. \end{aligned}$$

If $\Gamma_\varrho = \Gamma$, we denote

$$N(D) = \{u \mid u \in H_0^1(D) \cap H^2(D)\}.$$

It follows from the property of $H_0^1(\Omega)$ (see, for example, p. 62 of [24] or Corollary 6.2.43 of [16] or [29]) that $u = 0$ on ∂D for any $u \in H_0^1(\Omega)$ (Therefore, we always have that $u = 0$ on Γ for any $u \in K(D)$ or $N(D)$).

We shall also use the notation

$$\langle u, v \rangle^* = \int_D (\Delta_g u)(\Delta_g v) dR, \quad u, v \in K(D) \text{ or } K^d(D) \text{ or } N(D).$$

The bilinear functional $\langle u, v \rangle^*$ can be used as an inner product in each of the spaces $K(D)$, $K^d(D)$ and $N(D)$. In fact, $\langle u, v \rangle^*$ is a positive, symmetric, bilinear functional. In addition, if $\langle u, u \rangle^* = 0$, then $\Delta_g u = 0$ in D . In the case $u \in K(D)$ or $N(D)$, by applying the maximum principle, we have $u \equiv 0$ in D . In the case $u \in K^d(D)$, since $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_{00} , we find by Holmgren's uniqueness theorem (see, Corollary 5 of p. 39 in [37]) that $u \equiv 0$ in D . Closing $K(D)$, $K^d(D)$ and $N(D)$ with respect to the norm $\|u\|^* = \sqrt{\langle u, u \rangle^*}$, we get the Hilbert spaces $(\mathcal{K}, \|\cdot\|^*)$, $(\mathcal{K}^d, \|\cdot\|^*)$ and $(\mathcal{N}, \|\cdot\|^*)$, respectively.

Next, we consider two linear functionals

$$[u, v] = \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds$$

and

$$(3.1) \quad \langle u, v \rangle = \langle u, v \rangle^* + [u, v],$$

where $u, v \in K(D)$ or $u, v \in K^d(D)$ or $u, v \in N(D)$. It is clear that $\langle u, v \rangle$ is an inner product in each of the spaces $K(D)$, $K^d(D)$ and $N(D)$.

Lemma 3.1. *The norm*

$$\|u\|^\star = \sqrt{\langle u, u \rangle^\star}$$

and

$$\|u\| = \sqrt{\langle u, u \rangle}$$

are equivalent in $K(D)$, $K^d(D)$ and $N(D)$.

Proof. Obviously, $\|u\|^\star \leq \|u\|$ for all u in $K(D)$ or $K^d(D)$ or $N(D)$. In order to prove the equivalence of the two norms, we first consider the case in linear space $N(D)$. It suffices to show that $\|u\|$ is bounded when u belongs to the set

$$\mathfrak{M} = \{u \mid u \in N(D), \|u\|^\star \leq 1\}.$$

It follows from Lemma 2.1 that $\mathfrak{M}_\Gamma := \{\frac{\partial u}{\partial \nu} \mid u \in \mathfrak{M}\}$ is precompact in $L^2(\Gamma)$. This implies that there exists a constant $C > 0$ such that $\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 ds \leq C$ for all $u \in \mathfrak{M}$. Therefore, $[u, u] = \int_\Gamma \varrho \left(\frac{\partial u}{\partial \nu}\right)^2 ds$ is bounded in \mathfrak{M} , and so is $\|u\|^2 = \langle u, u \rangle^\star + [u, u]$. Similarly, applying Lemmas 2.1, 2.2 we can prove the corresponding results for the spaces $K(D)$ and $K^d(D)$. \square

From Lemmas 2.1, 2.2, it follows that

$$|[u, u]| = \left| \int_{\Gamma_e} \varrho \left(\frac{\partial u}{\partial \nu}\right)^2 ds \right| \leq C \langle u, u \rangle^\star \text{ for all } u \text{ in } K(D) \text{ or } K^d(D) \text{ or } N(D).$$

Therefore, $[u, v]$ is a bounded, symmetric, bilinear functional in $(K(D), \langle \cdot, \cdot \rangle^\star)$, $(K^d(D), \langle \cdot, \cdot \rangle^\star)$ and $(N(D), \langle \cdot, \cdot \rangle^\star)$. Since it is densely defined in $(\mathcal{K}, \langle \cdot, \cdot \rangle^\star)$, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^\star)$ and $(\mathcal{N}, \langle \cdot, \cdot \rangle^\star)$, respectively, it can immediately be extended to $(\mathcal{K}, \langle \cdot, \cdot \rangle^\star)$, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^\star)$ and $(\mathcal{N}, \langle \cdot, \cdot \rangle^\star)$. We still use $[u, v]$ to express the extended functional. Then there is a bounded linear transformation $G_{\mathcal{K}}^{(\star)}$ of $(\mathcal{K}, \langle \cdot, \cdot \rangle^\star)$ into $(\mathcal{K}, \langle \cdot, \cdot \rangle^\star)$ (respectively, $G_{\mathcal{K}^d}^{(\star)}$ of $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^\star)$ into $(\mathcal{K}^d, \langle \cdot, \cdot \rangle^\star)$, $G_{\mathcal{N}}^{(\star)}$ of $(\mathcal{N}, \langle \cdot, \cdot \rangle^\star)$ into $(\mathcal{N}, \langle \cdot, \cdot \rangle^\star)$) such that

$$(3.2) \quad [u, v] = \langle G_{\mathcal{K}}^{(\star)} u, v \rangle^\star \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}$$

(respectively,

$$(3.3) \quad [u, v] = \langle G_{\mathcal{K}^d}^{(\star)} u, v \rangle^\star \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}^d,$$

$$(3.4) \quad [u, v] = \langle G_{\mathcal{N}}^{(\star)} u, v \rangle^\star \quad \text{for all } u \text{ and } v \text{ in } \mathcal{N}.$$

Lemma 3.2. *The transformations $G_{\mathcal{K}}^{(\star)}$, $G_{\mathcal{K}^d}^{(\star)}$ and $G_{\mathcal{N}}^{(\star)}$ are self-adjoint and compact.*

Proof. Since $[u, v]$ is symmetric, we immediately get that the transformation $G_{\mathcal{K}}^{(*)}$, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$ are all self-adjoint. For the compactness, we only discuss the case for the transformation $G_{\mathcal{K}}^{(*)}$. It suffices to show (see, p.204 of [38]):

From every sequence $\{u_m\}$ in $K(D)$ which is bounded

$$(3.5) \quad \|u_m\|^* \leq \text{constant}, \quad m = 1, 2, 3, \dots,$$

we can pick out a subsequence $\{u_{m'}\}$ such that

$$(3.6) \quad \langle G_{\mathcal{K}}^{(*)}(u_{m'} - u_{l'}), (u_{m'} - u_{l'}) \rangle^* \rightarrow 0 \quad \text{when } m', l' \rightarrow \infty.$$

Applying Lemma 2.1 with the aid of (3.5), we find that the sequence $\{\frac{\partial u_m}{\partial \nu}\}$ is precompact in $L^2(\Gamma_\varrho)$, so that there is a subsequence $\{u_{m'}\}$ such that

$$\int_{\Gamma_\varrho} \left(\frac{\partial(u_{m'} - u_{l'})}{\partial \nu} \right)^2 ds \rightarrow 0 \quad \text{as } m', l' \rightarrow \infty.$$

Therefore

$$[u_{m'} - u_{l'}, u_{m'} - u_{l'}] = \int_{\Gamma_\varrho} \varrho \left(\frac{\partial(u_{m'} - u_{l'})}{\partial \nu} \right)^2 ds \rightarrow 0 \quad \text{as } m', l' \rightarrow \infty,$$

which implies (3.6). This proves the compactness of $G_{\mathcal{K}}^{(*)}$. \square

Except for the transformations $G_{\mathcal{K}}^{(*)}$, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$, we need introduce corresponding transformations $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ by the inner product $\langle \cdot, \cdot \rangle$. Since

$$(3.7) \quad 0 \leq [u, u] \leq \langle u, u \rangle \quad \text{for all } u \text{ in } K(D) \text{ or } K^d(D) \text{ or } N(D),$$

there is a bounded linear self-adjoint transformation $G_{\mathcal{K}}$ of $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ (respectively, $G_{\mathcal{K}^d}$ of $(\mathcal{K}^d, \langle \cdot, \cdot \rangle)$, $G_{\mathcal{N}}$ of $(\mathcal{N}, \langle \cdot, \cdot \rangle)$) such that

$$(3.8) \quad [u, v] = \langle G_{\mathcal{K}} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}$$

(respectively,

$$(3.9) \quad [u, v] = \langle G_{\mathcal{K}^d} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}^d,$$

$$(3.10) \quad [u, v] = \langle G_{\mathcal{N}} u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{N}).$$

Lemma 3.3. *The transformations $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ are positive and compact.*

Proof. From $[u, u] \geq 0$ for any $u \in \mathcal{K}$ or \mathcal{K}^d or \mathcal{N} , we immediately know that $G_{\mathcal{K}}$, $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$ are positive. The proof of the compactness is completely similar to that of Lemma 3.2. \square

It follows from Lemma 3.3 that $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}$) has only non-negative eigenvalues and that the positive eigenvalues form an enumerable sequence $\{\mu_{\mathcal{K}}\}$ (respectively, $\{\mu_{\mathcal{K}^d}\}$, $\{\mu_{\mathcal{N}}\}$) with 0 as the only limit point.

Theorem 3.4. *The transformations $G_{\mathcal{K}}^{(*)}$ and $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}^{(*)}$ and $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}^{(*)}$ and $G_{\mathcal{N}}$) have the same eigenfunctions. If $\mu_{\mathcal{K}}^*$ and $\mu_{\mathcal{K}}$ (respectively, $\mu_{\mathcal{K}^d}^*$ and $\mu_{\mathcal{K}^d}$, $\mu_{\mathcal{N}}^*$ and $\mu_{\mathcal{N}}$) are eigenvalues corresponding to the same eigenfunction we have*

$$(3.11) \quad \mu_{\mathcal{K}} = \frac{\mu_{\mathcal{K}}^*}{1 + \mu_{\mathcal{K}}^*}$$

(respectively,

$$(3.12) \quad \mu_{\mathcal{K}^d} = \frac{\mu_{\mathcal{K}^d}^*}{1 + \mu_{\mathcal{K}^d}^*},$$

$$(3.13) \quad \mu_{\mathcal{N}} = \frac{\mu_{\mathcal{N}}^*}{1 + \mu_{\mathcal{N}}^*}.$$

Proof. We only prove the case for the $G_{\mathcal{K}}$ (a similar argument will work for $G_{\mathcal{K}^d}$ and $G_{\mathcal{N}}$). Since $G_{\mathcal{K}}^{(*)}$ is positive, we can easily conclude that the inverse $(1 + G_{\mathcal{K}}^{(*)})^{-1}$ exists and is a bounded self-adjoint transformation. By virtue of (3.2), (3.8) and (3.1), we have

$$(3.14) \quad \begin{aligned} \langle G_{\mathcal{K}}^{(*)} u, v \rangle^* &= [u, v] = \langle G_{\mathcal{K}} u, v \rangle \\ &= \langle G_{\mathcal{K}} u, v \rangle^* + \langle G_{\mathcal{K}} G_{\mathcal{K}}^{(*)} u, v \rangle^*, \quad (u, v \in \mathcal{K}). \end{aligned}$$

It follows that

$$(3.15) \quad G_{\mathcal{K}} = G_{\mathcal{K}}^{(*)} (1 + G_{\mathcal{K}}^{(*)})^{-1},$$

from which the desired result follows immediately. \square

Proposition 3.5. *Let u and v be two eigenfunctions in $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ (respectively, $(\mathcal{K}^d, \langle \cdot, \cdot \rangle)$, $(\mathcal{N}, \langle \cdot, \cdot \rangle)$) of the transformation $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}$) at least one of which corresponds to a non-vanishing eigenvalue. Then u and v are orthogonal if and only if the $\frac{\partial u}{\partial \nu}|_{\Gamma_e}$ and $\frac{\partial v}{\partial \nu}|_{\Gamma_e}$ are orthogonal in $L^2(\Gamma_e)$, that is,*

$$(3.16) \quad [u, v] = \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds = 0.$$

Proof. Without loss of generality, we suppose that u is the eigenfunction corresponding to the eigenvalue $\mu \neq 0$. Then

$$[u, v] = \langle G_{\mathcal{K}} u, v \rangle = \mu \langle u, v \rangle,$$

which implies the desired result. \square

We can now prove

Theorem 3.6. *Let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary Γ . Assume that Γ_{00} is an $(n-1)$ -dimensional surface in $\Gamma - \bar{\Gamma}_e$. If u is an eigenfunction of the transformations $G_{\mathcal{K}}^{(*)}$ or $G_{\mathcal{N}}^{(*)}$ with eigenvalue $\mu^* \neq 0$, then u has derivatives of any order in D and is such that*

$$(3.17) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \quad \Delta_g u = 0 & \text{on } \Gamma - (\Gamma_e \cup \Gamma_{00}), \\ \Delta_g u + \gamma \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_e, \quad \text{with } \gamma = \frac{1}{\mu^*}. \end{cases}$$

Proof. Let $\{u_j\}$ be a sequence of functions in $K(D)$ such that $\|u_j - u\|^* \rightarrow 0$ as $j \rightarrow \infty$. We first claim that

$$(3.18) \quad u_j \rightarrow u \text{ in } L^2(D).$$

In fact, since $u_j \in H_0^1(D) \cap H^2(D)$, it follows from (2.7) and (2.8) that

$$(3.19) \quad \int_D |u_j - u_l|^2 dR \leq \frac{1}{\lambda_1(D)} \int_D |\nabla_g(u_j - u_l)|^2 dR \quad \text{for any } j \text{ and } l,$$

$$(3.20) \quad \int_D |\nabla_g(u_j - u_l)|^2 dR \leq \frac{1}{\Lambda_1(D)} \int_D |\Delta_g(u_j - u_l)|^2 dR \quad \text{for any } j \text{ and } l,$$

where $\lambda_1(D)$ and $\Lambda_1(D)$ are the first Dirichlet and buckling eigenvalues for D , respectively. Since $\int_D |\Delta_g(u_j - u_l)|^2 dR \rightarrow 0$ as $j, l \rightarrow +\infty$, we find by (3.19) and (3.20) that $\int_D |u_j - u_l|^2 dR \rightarrow 0$ as $j, l \rightarrow +\infty$. Therefore the claim is proved.

For any point p in D , let U be a coordinate neighborhood of p , and let $E \ni p$ be a bounded domain with smooth boundary such that $\bar{E} \subset U \cap D$. Let f be a function in $C_0^4(E)$. Then, by Green's formula (see, for example, p. 6 of [4]), we have

$$\langle u_n, f \rangle^* = \int_E (\Delta_g u_n)(\Delta_g f) dR = \int_E u_n(\Delta_g^2 f) dR,$$

so that

$$(3.21) \quad \langle u, f \rangle^* = \int_E u(\Delta_g^2 f) dR.$$

Now by assumption, $G_{\mathcal{K}}^{(*)}u = \mu^*u$ with $\mu^* \neq 0$ and $\frac{\partial f}{\partial \nu} = 0$ on Γ , and hence we have

$$(3.22) \quad \mu^* \langle u, f \rangle^* = \langle G_{\mathcal{K}}^{(*)}u, f \rangle^* = [u, f] = 0.$$

Since p is arbitrary in D , it follows from (3.21) and (3.22) that

$$\int_D u(\Delta_g^2 f) dR = 0 \quad \text{for all } f \in C_0^4(D).$$

By applying Green's formula again, we get

$$(3.23) \quad \int_D (\Delta_g u)(\Delta_g f) dR = 0 \quad \text{for all } f \in C_0^4(D),$$

i.e., u is a weak solution of $\Delta_g^2 u = 0$ in D (see [13]). It follows from the interior regularity of elliptic equations that $u \in C^\infty(D)$, and in the classic sense

$$(3.24) \quad \Delta_g^2 u = 0 \quad \text{in } D.$$

In exactly the same way, the corresponding result can be proved for $G_{\mathcal{N}}^{(*)}$.

Next, suppose that ϱ is continuous. That the boundary conditions of (3.17) hold follows from Lemma 2.1 and Green's formula. In fact, if

$$G_{\mathcal{K}}^{(*)}u = \mu^*u,$$

then $u|_{\Gamma} = 0$ and $\frac{\partial u}{\partial \nu}|_{\Gamma_{00}} = 0$, and that

$$\int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds = \mu^* \int_D (\Delta_g u)(\Delta_g v) dR \quad \text{for all } v \in \mathcal{K}(D).$$

By this and Green's formula (see, p. 114-120 of [24], [26] and [10]), we obtain that

$$\frac{1}{\mu^*} \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds = \int_D (\Delta_g^2 u)v dR - \int_{\Gamma} (\Delta_g u) \frac{\partial v}{\partial \nu} ds + \int_{\Gamma} \frac{\partial(\Delta_g u)}{\partial \nu} v ds.$$

for all $v \in \mathcal{K}(D)$, where $\frac{\partial(\Delta_g u)}{\partial \nu} \in H^{-3/2}(\Gamma)$ (see [2]). Thus

$$(3.25) \quad \int_D (\Delta_g^2 u) v \, dR - \int_{\Gamma_\varrho} \left(\Delta_g u + \frac{1}{\mu^\star} \varrho \frac{\partial u}{\partial \nu} \right) \frac{\partial v}{\partial \nu} \, ds$$

$$(3.26) \quad + \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_{00})} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds + \int_\Gamma \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds = 0$$

for all $v \in \mathcal{K}(D)$. Note that $v|_\Gamma = 0$ and $\frac{\partial v}{\partial \nu}|_{\Gamma_{00}} = 0$, and that $\frac{\partial v}{\partial \nu}|_{\Gamma_\varrho}$ and $\frac{\partial v}{\partial \nu}|_{\Gamma - (\Gamma_\varrho \cup \Gamma_{00})}$ run throughout space $L^2(\Gamma_\varrho)$ and $L^2(\Gamma - (\Gamma_\varrho \cup \Gamma_{00}))$, respectively, when v runs throughout space $K(D)$. This implies that

$$\Delta_g u = 0 \text{ on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \text{ and } \Delta_g u + \frac{1}{\mu^\star} \varrho \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho.$$

Therefore, (3.17) holds. In a similar way, we can prove the desired result for $G_{\mathcal{N}}$. \square

Theorem 3.7. *Let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let $D \subset (\mathcal{M}, g)$ be a bounded domain with piecewise smooth boundary Γ . Assume that Γ_{00} is a real analytic $(n-1)$ -dimensional surface in $\Gamma - \bar{\Gamma}_\varrho$. If u is an eigenfunction of the transformations $G_{\mathcal{K}^d}^{(\star)}$ with eigenvalue $\mu^\star \neq 0$, then u has derivatives of order in D and is such that*

$$(3.27) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \\ \Delta_g u = 0 \text{ and } \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \\ \Delta_g u + \kappa \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \quad \text{with } \kappa = \frac{1}{\mu^\star}. \end{cases}$$

Proof. If $G_{\mathcal{K}^d}^{(\star)} u = \mu^\star u$, then we have that $u = 0$ on Γ_ϱ and $u = \frac{\partial u}{\partial \nu} = 0$ on Γ_{00} , and that

$$(3.28) \quad \int_{\Gamma_\varrho} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \, ds = \mu^\star \int_D (\Delta_g u)(\Delta_g v) \, dR \quad \text{for all } v \in \mathcal{K}^d(D).$$

Applying Green's formula on the right-hand side of (3.28), we get that

$$(3.29) \quad \begin{aligned} & \int_D (\Delta_g^2 u) v \, dR + \int_{\partial D} \frac{\partial(\Delta_g u)}{\partial \nu} v \, ds - \int_{\Gamma - (\Gamma_\varrho \cup \Gamma_{00})} (\Delta_g u) \frac{\partial v}{\partial \nu} \, ds \\ & - \int_{\Gamma_\varrho} \left(\Delta_g u + \frac{1}{\mu^\star} \varrho \frac{\partial u}{\partial \nu} \right) \frac{\partial v}{\partial \nu} \, ds = 0 \quad \text{for all } v \in K^d(D). \end{aligned}$$

By taking all $v \in C_0^\infty(D)$, we obtain $\Delta_g^2 u = 0$ in D . Note that $v|_{\Gamma_\varrho} = 0$ and $v|_{\Gamma_{00}} = \frac{\partial v}{\partial \nu}|_{\Gamma_{00}} = 0$, and that $v|_{\Gamma - (\Gamma_\varrho \cup \Gamma_{00})}$ and $\frac{\partial v}{\partial \nu}|_{\Gamma - \Gamma_{00}}$ run throughout the spaces $L^2(\Gamma - (\Gamma_\varrho \cup \Gamma_{00}))$ and $L^2(\Gamma - \Gamma_{00})$, respectively, when v runs throughout the space $K^d(D)$. Thus we have

$$\begin{aligned} \Delta_g u &= 0 \text{ and } \frac{\partial(\Delta_g u)}{\partial \nu} = 0 \text{ on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \\ \Delta_g u + \frac{1}{\mu^\star} \varrho \frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma_\varrho. \end{aligned} \quad \square$$

Theorem 3.8. *Let (\mathcal{M}, g) , D and Γ_{00} be as in Theorem 3.7. Assume that ς_k and κ_k are the k -th Steklov eigenvalues of the following problems:*

$$(3.30) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \\ \Delta_g u + \varsigma \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho \end{cases}$$

and

$$(3.31) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_\varrho, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \\ \Delta_g u = 0 \text{ and } \frac{\partial(\Delta_g u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \\ \Delta_g u + \kappa \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho, \end{cases}$$

respectively. Then $\varsigma_k \leq \kappa_k$ for all $k \geq 1$.

Proof. For $0 < \alpha < 1$, let $u_k = u_k(\alpha, x)$ be the normalized eigenfunction corresponding to the k -th Steklov eigenvalue λ_k for the following problem:

$$(3.32) \quad \begin{cases} \Delta_g^2 u_k = 0 & \text{in } D, \\ u_k = 0 & \text{on } \Gamma_\varrho, \\ u_k = \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \\ \alpha \Delta_g u_k + (1 - \alpha) \frac{\partial u_k}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta_g u_k)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}), \\ \Delta_g u_k + \lambda \varrho \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \Gamma_\varrho. \end{cases}$$

It is easy to verify (cf, p. 410 or Theorem 9 of p. 419 in [6]) that the k -th Steklov eigenvalue $\lambda_k = \lambda_k(\alpha)$ is continuous on the closed interval $[0, 1]$ and differentiable in the open interval $(0, 1)$, and that $u_k(\alpha, x)$ is also differentiable with respect to α in $(0, 1)$ (see, [12]). We will denote by $'$ the derivative with respect to α . Then

$$(3.33) \quad \begin{cases} \Delta_g^2 u'_k = 0 & \text{in } D, \\ u'_k = 0 & \text{on } \Gamma_\varrho, \\ u'_k = \frac{\partial u'_k}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \\ \Delta_g u_k + \alpha \Delta_g u'_k - \frac{\partial u_k}{\partial \nu} + (1 - \alpha) \frac{\partial u'_k}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta_g u'_k)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho \cup \Gamma_{00}) \\ \Delta_g u'_k + \lambda' \varrho \frac{\partial u_k}{\partial \nu} + \lambda \varrho \frac{\partial u'_k}{\partial \nu} = 0 & \text{on } \Gamma_\varrho. \end{cases}$$

Multiplying (3.33) by u_k , integrating the product over D , and then applying Green's formula, we get

$$\begin{aligned}
0 &= \int_D (\Delta_g^2 u'_k) u_k dR = \int_D (\Delta_g^2 u_k) u'_k dR - \int_{\partial D} (\Delta_g u_k) \frac{\partial u'_k}{\partial \nu} ds \\
&\quad + \int_{\partial D} u'_k \frac{\partial (\Delta_g u_k)}{\partial \nu} ds - \int_{\partial D} u_k \frac{\partial (\Delta_g u'_k)}{\partial \nu} ds + \int_{\partial D} (\Delta_g u'_k) \frac{\partial u_k}{\partial \nu} ds \\
&= - \left[\int_{\Gamma_e} (\Delta_g u_k) \frac{\partial u'_k}{\partial \nu} ds + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} (\Delta_g u_k) \frac{\partial u'_k}{\partial \nu} ds \right] \\
&\quad + \left[\int_{\Gamma_e} (\Delta_g u'_k) \frac{\partial u_k}{\partial \nu} ds + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} (\Delta_g u'_k) \frac{\partial u_k}{\partial \nu} ds \right] \\
&= \left[\int_{\Gamma_e} \left(\lambda' \varrho \frac{\partial u_k}{\partial \nu} \right) \frac{\partial u'_k}{\partial \nu} ds + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} \left(\frac{1-\alpha}{\alpha} \frac{\partial u_k}{\partial \nu} \right) \frac{\partial u'_k}{\partial \nu} ds \right] \\
&\quad + \int_{\Gamma_e} \left(-\lambda' \varrho \frac{\partial u_k}{\partial \nu} - \lambda \varrho \frac{\partial u'_k}{\partial \nu} \right) \frac{\partial u_k}{\partial \nu} ds \\
&\quad + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} \left(-\frac{1}{\alpha} \Delta_g u_k + \frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} - \frac{1-\alpha}{\alpha} \frac{\partial u'_k}{\partial \nu} \right) \frac{\partial u_k}{\partial \nu} ds \\
&= -\lambda' \int_{\Gamma_e} \varrho \left(\frac{\partial u_k}{\partial \nu} \right)^2 ds + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} \left[\left(\frac{1-\alpha}{\alpha^2} \right) \frac{\partial u_k}{\partial \nu} + \frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right] \frac{\partial u_k}{\partial \nu} ds \\
&= -\lambda' \int_{\Gamma_e} \varrho \left(\frac{\partial u_k}{\partial \nu} \right)^2 ds + \int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} \left(\frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right)^2 ds,
\end{aligned}$$

i.e.,

$$\lambda'_k(\alpha) = \frac{\int_{\Gamma - (\Gamma_e \cup \Gamma_{00})} \left(\frac{1}{\alpha} \frac{\partial u_k}{\partial \nu} \right)^2 ds}{\int_{\Gamma_e} \varrho \left(\frac{\partial u_k}{\partial \nu} \right)^2 ds} > 0 \quad \text{for all } 0 < \alpha < 1.$$

This implies that λ_k is increasing with respect to α in $(0, 1)$. Note that if we change the α from 0 to 1, each individual Steklov eigenvalue λ_k increase monotonically from the value ς_k which is the k -th Steklov eigenvalue of (3.30) to the value κ_k which is the k -th Steklov eigenvalue (3.31). Thus, we have that $\varsigma_k \leq \kappa_k$ for all k . \square

Conversely, the following proposition shows that a sufficiently smooth function satisfying (3.17) (respectively, (3.27)) is an eigenfunction of $G_{\mathcal{K}}^{(*)}$ or $G_{\mathcal{N}}^{(*)}$ (respectively, $G_{\mathcal{K}^d}^{(*)}$).

Proposition 3.9. *Let \bar{D} be bounded domain with piecewise smooth boundary. Assume that u belongs to $C^4(\bar{D})$.*

a) *If $\Gamma_e \neq \Gamma$ and u satisfies (3.17), then $u \in \mathcal{K}$ and u is an eigenfunction of $G_{\mathcal{K}}^{(*)}$ with the eigenvalue $\mu^* = \gamma^{-1}$,*

$$(3.34) \quad G_{\mathcal{K}}^{(*)} u = \gamma^{-1} u.$$

b) *If $\Gamma_e \neq \Gamma$ and u satisfies (3.27), then $u \in \mathcal{K}^d$ and u is an eigenfunction of $G_{\mathcal{K}^d}^{(*)}$ with the eigenvalue $\mu^* = \kappa^{-1}$,*

$$(3.35) \quad G_{\mathcal{K}^d}^{(*)} u = \kappa^{-1} u.$$

c) If $\Gamma_\varrho = \Gamma$ and u satisfies (3.17), then $u \in \mathcal{N}$ and u is an eigenfunction of $G_{\mathcal{N}}^{(\star)}$ with the eigenvalue $\mu^\star = \gamma^{-1}$,

$$(3.36) \quad G_{\mathcal{N}}^{(\star)} u = \gamma^{-1} u.$$

Proof. i) $\Gamma_\varrho \neq \Gamma$. We claim that there is no eigenvalue $\gamma = 0$. Suppose by contradiction that there is a function u in $C^4(\bar{D})$ satisfying

$$(3.37) \quad \begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{00}, \end{cases} \quad u = 0 \text{ on } \Gamma, \quad \text{and } \Delta_g u = 0 \text{ on } \Gamma - \Gamma_{00}.$$

By multiplying the above equation by u , integrating the result over D , and using Green's formula, we derive

$$\begin{aligned} 0 &= \int_D u(\Delta_g^2 u) dR = \int_D |\Delta_g u|^2 dR - \int_\Gamma u \frac{\partial(\Delta_g u)}{\partial \nu} ds \\ &\quad + \int_\Gamma (\Delta_g u) \frac{\partial u}{\partial \nu} ds = \int_D |\Delta_g u|^2 dR. \end{aligned}$$

This implies that $\Delta_g u = 0$ in D . Since $u = 0$ on Γ , by the maximum principle we get that $u = 0$ in D . The claim is proved.

In view of assumptions, we see that $u \in \mathcal{K}$. By (3.17) and Green's formula, it follows that for an arbitrary $v \in K(D)$

$$\begin{aligned} \langle G_{\mathcal{K}}^{(\star)} u, v \rangle^\star &= [u, v] = \int_{\Gamma_\varrho} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds \\ &= -\gamma^{-1} \int_{\Gamma_\varrho} (\Delta_g u) \frac{\partial v}{\partial \nu} ds = -\gamma^{-1} \int_\Gamma (\Delta_g u) \frac{\partial v}{\partial \nu} ds \\ &= -\gamma^{-1} \left[\int_\Gamma \frac{\partial(\Delta_g u)}{\partial \nu} v ds - \int_D (\Delta_g u)(\Delta_g v) dR + \int_D v(\Delta_g^2 u) dR \right] \\ &= \gamma^{-1} \int_D (\Delta_g u)(\Delta_g v) dR = \gamma^{-1} \langle u, v \rangle^\star. \end{aligned}$$

Therefore,

$$\langle G_{\mathcal{K}}^{(\star)} u - \gamma^{-1} u, v \rangle^\star = 0 \quad \text{for all } v \in K(D),$$

which implies (3.34). By a similar way, we can prove b).

ii) $\Gamma_\varrho = \Gamma$. We claim that there is no eigenvalue $\gamma = 0$. If it is not this case, then there is a function u in $C^4(\bar{D})$ satisfying

$$\begin{cases} \Delta_g^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma, \\ \Delta_g u = 0 & \text{on } \Gamma. \end{cases}$$

Setting $v := \Delta_g u$ in D , we get

$$\begin{cases} \Delta_g v = 0 & \text{in } D, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

By the maximum principle it follows that $v = 0$ in D . Thus, we have

$$\begin{cases} \Delta_g u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

so that $u = 0$ in D .

Now, if u is a solution of (3.17) with eigenvalue $\gamma > 0$, proceeding as in a), we can prove that $u \in \mathcal{N}$ and (3.36) holds. \square

Remark 3.10. Each of transformations $G_{\mathcal{K}}^*$, $G_{\mathcal{K}^d}^*$ and $G_{\mathcal{N}}^*$ corresponds to a biharmonic Steklov problem given by the quadratic forms

$$\langle u, u \rangle^* = \int_D |\Delta_g u|^2 dR$$

and

$$[u, u] = \int_{\Gamma_\varrho} \varrho \left(\frac{\partial u}{\partial \nu} \right)^2 ds$$

and the function classes of \mathcal{K}^* , \mathcal{K}^{d*} and \mathcal{N}^* , respectively. The eigenvalues γ_k and κ_k of these biharmonic Steklov problems are given by

$$(3.38) \quad \gamma_k \text{ and } \kappa_k = 1/\mu_k^* \quad k = 1, 2, 3, \dots$$

Since 0 is the only limit point of μ_k^* , the only possible limit points of γ_k and κ_k are $+\infty$.

4. BIHARMONIC STEKLOV EIGENVALUES ON AN n -DIMENSIONAL RECTANGULAR PARALLELEPIPED

Let $D = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i, i = 1, \dots, n\}$ with boundary Γ , and let $\Gamma_\varrho^+ = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = 0\}$. Let $\Gamma^{l_n} = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = l_n\}$. Our first purpose, in this section, is to discuss the biharmonic Steklov eigenvalue problem on n -dimensional rectangular parallelepiped D :

$$(4.1) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \quad \Delta u = 0 & \text{on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \Delta u + \gamma \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 & \text{on } \Gamma_\varrho^+. \end{cases}$$

We first consider the special solution of (4.1) which has the following form:

$$u = X(x_1, \dots, x_{n-1}) Y(x_n).$$

Since

$$\begin{aligned} \Delta u &= (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y(x_n) + 2 \nabla X(x_1, \dots, x_{n-1}) \cdot \nabla Y(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''(x_n) = (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''(x_n) \end{aligned}$$

and

$$\begin{aligned} \Delta^2 u &= (\Delta_{n-1}^2 X(x_1, \dots, x_{n-1})) Y(x_n) + 2 (\Delta_{n-1} X(x_1, \dots, x_{n-1})) Y''(x_n) \\ &\quad + (X(x_1, \dots, x_{n-1})) Y''''(x_n), \end{aligned}$$

where

$$\Delta_{n-1} X(x_1, \dots, x_{n-1}) = \sum_{i=1}^{n-1} \frac{\partial^2 X}{\partial x_i^2},$$

we find by $\Delta^2 u = 0$ that

$$\begin{aligned} & (\Delta_{n-1}^2 X(x_1, \dots, x_{n-1}))Y(x_n) + 2(\Delta_{n-1} X(x_1, \dots, x_{n-1}))Y''(x_n) \\ & + (X(x_1, \dots, x_{n-1}))Y''''(x_n) = 0, \end{aligned}$$

so that

$$(4.2) \quad \frac{\Delta_{n-1}^2 X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} + 2 \frac{\Delta_{n-1} X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} \frac{Y''(x_n)}{Y(x_n)} + \frac{Y''''(x_n)}{Y(x_n)} = 0.$$

Differentiating (4.2) with respect to x_n , we obtain that

$$2 \frac{\Delta_{n-1} X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} \left[\frac{Y''(x_n)}{Y(x_n)} \right]' + \left[\frac{Y''''(x_n)}{Y(x_n)} \right]' = 0.$$

The above equation holds if and only if

$$(4.3) \quad \frac{\Delta_{n-1} X(x_1, \dots, x_{n-1})}{X(x_1, \dots, x_{n-1})} = - \frac{\left[\frac{Y''''(x_n)}{Y(x_n)} \right]'}{2 \left[\frac{Y''(x_n)}{Y(x_n)} \right]'} = -\eta^2,$$

where η^2 is a constant. Therefore, we have that

$$(4.4) \quad \Delta_{n-1} X(x_1, \dots, x_{n-1}) + \eta^2 X(x_1, \dots, x_{n-1}) = 0$$

and

$$\left[\frac{Y''''(x_n)}{Y(x_n)} \right]' - 2\eta^2 \left[\frac{Y''(x_n)}{Y(x_n)} \right]' = 0.$$

From (4.4), we get

$$(4.5) \quad \Delta_{n-1}^2 X(x_1, \dots, x_{n-1}) = -\eta^2 \Delta_{n-1} X(x_1, \dots, x_{n-1}) = \eta^4 X(x_1, \dots, x_{n-1}).$$

Substituting this in (4.2), we obtain the following equation

$$(4.6) \quad Y''''(x_n) - 2\eta^2 Y''(x_n) + \eta^4 Y(x_n) = 0.$$

It is easy to verify that the general solutions of (4.6) have the form:

$$(4.7) \quad Y(x_n) = A \cosh \eta x_n + B \sinh \eta x_n + C x_n \cosh \eta x_n + D x_n \sinh \eta x_n.$$

By setting $Y(0) = Y(l_n) = 0$, $Y'(0) = 1$, $Y'(l_n) = 0$, we get

$$(4.8) \quad \begin{aligned} Y(x_n) = & \left(\frac{-\eta l_n^2}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \sinh \eta x_n + \left(\frac{\sinh^2 \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \cosh \eta x_n \\ & + \left(\frac{\eta l_n - (\sinh \eta l_n) \cosh \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \sinh \eta x_n. \end{aligned}$$

It is well-known that for the Dirichlet eigenvalue problem

$$(4.9) \quad \begin{cases} \Delta_{n-1} X(x_1, \dots, x_{n-1}) + \eta^2 X(x_1, \dots, x_{n-1}) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\{(x_1, \dots, x_{n-1}) | 0 \leq x_i \leq l_i, i = 1, \dots, n-1\}, \end{cases}$$

there exist the eigenfunctions

$$(4.10) \quad X(x_1, \dots, x_{n-1}) = c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right),$$

which correspond to the eigenvalues

$$\eta^2 = \sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2, \quad \text{where } m_i = 1, 2, 3, \dots.$$

Therefore,

$$\begin{aligned}
 (4.11) \quad u &= (X(x_1, \dots, x_{n-1}))Y(x_n) \\
 &= c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) \left[\left(\frac{-\eta l_n^2}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \sinh \eta x_n \right. \\
 &\quad + \left(\frac{\sinh^2 \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \cosh \eta x_n \\
 &\quad \left. + \left(\frac{\eta l_n - (\sinh \eta l_n) \cosh \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) x_n \sinh \eta x_n \right].
 \end{aligned}$$

Since

$$Y''(0) = 2\eta \left(\frac{\eta l_n - (\sinh \eta l_n) \cosh \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \quad \text{and} \quad Y'(0) = 1,$$

we obtain

$$\begin{aligned}
 (\Delta u)|_{x_n=0} &= (\Delta_{n-1} X(x_1, \dots, x_{n-1}))Y(0) + (X(x_1, \dots, x_{n-1}))Y''(0) \\
 &= 2\eta \left(\frac{\eta l_n - (\sinh \eta l_n) \cosh \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) X(x_1, \dots, x_{n-1}),
 \end{aligned}$$

and

$$\frac{\partial u}{\partial \nu} \Big|_{\Gamma_\rho^+} = X(x_1, \dots, x_{n-1}),$$

so that

$$\Delta u + \gamma \varrho \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_\varrho^+$$

with

$$\gamma = \frac{2\eta l_n}{\varrho l_n} \left(\frac{(\sinh \eta l_n) \cosh \eta l_n - \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right).$$

Our second purpose is to discuss the biharmonic Steklov eigenvalue problem on the n -dimensional rectangular parallelepiped D :

$$(4.12) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u = 0 \text{ on } \Gamma_\varrho^+, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = 0 \text{ on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \Delta u + \varsigma \varrho \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

Similarly, (4.12) has the special solution $u = (X(x_1, \dots, x_{n-1}))Z(x_n)$ with $Z(x_n)$ having form (4.7). According to the boundary conditions of (4.12), we get that the problem (4.12) has the solutions

$$\begin{aligned}
 u(x) &= u(x_1, \dots, x_n) \\
 &= c \left(\cos \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\cos \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Z(x_n),
 \end{aligned}$$

where m_1, \dots, m_{n-1} are whole numbers, and $Z(x_n)$ is given by

$$\begin{aligned}
 (4.13) \quad Z(x_n) &= \left(\frac{-\beta l_n^2}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right) \sinh \beta x_n + \left(\frac{\sinh^2 \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right) x_n \cosh \beta x_n \\
 &\quad + \left(\frac{\beta l_n - (\sinh \beta l_n) \cosh \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right) x_n \sinh \beta x_n,
 \end{aligned}$$

$\beta = [\sum_{i=1}^{n-1} (m_i \pi / l_i)^2]^{1/2}$ with $\sum_{i=1}^{n-1} m_i \neq 0$. Since $\frac{\partial u}{\partial \nu}|_{\Gamma_\rho^+} = X(x_1, \dots, x_{n-1})$, $(\Delta u)|_{x_n=0} = (X(x_1, \dots, x_{n-1}))Z''(0)$ and $Z''(0) = 2\eta \left(\frac{\beta l_n - (\sinh \beta l_n) \cosh \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right)$, we get $\Delta u + \varsigma \varrho \frac{\partial u}{\partial \nu} = 0$ on Γ_ϱ^+ , where

$$\varsigma = \frac{2\beta l_n}{\varrho l_n} \left(\frac{(\sinh \beta l_n) \cosh \beta l_n - \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right).$$

5. ASYMPTOTIC DISTRIBUTION OF EIGENVALUES ON SPECIAL DOMAINS

5.1. Counting function $A(\tau)$.

In order to obtain our asymptotic formula, it is an effective way to investigate the distribution of the eigenvalues of the transformation $G_{\mathcal{K}}$ (respectively, $G_{\mathcal{K}^d}$, $G_{\mathcal{N}}$) instead of the transformations $G_{\mathcal{K}}^{(*)}$ (respectively, $G_{\mathcal{K}^d}^{(*)}$, $G_{\mathcal{N}}^{(*)}$). It follows from (3.11)–(3.13) and (3.38) we obtain

$$(5.1) \quad \mu_k = (1 + \lambda_k)^{-1}, \quad k = 1, 2, 3, \dots,$$

where μ_k denote the k -th eigenvalue of $G_{\mathcal{K}}$ or $G_{\mathcal{K}^d}$ or $G_{\mathcal{N}}$, and $\frac{1}{\lambda_k}$ is the k -th eigenvalue of $G_{\mathcal{K}}^{(*)}$ or $G_{\mathcal{K}^d}^{(*)}$ or $G_{\mathcal{N}}^{(*)}$ (More precisely, $\lambda_k = \gamma_k$ for $G_{\mathcal{K}}^{(*)}$ and $G_{\mathcal{N}}^{(*)}$, and $\lambda_k = \kappa_k$ for $G_{\mathcal{K}^d}^{(*)}$.) Since $A(\tau) = \sum_{\lambda_k \leq \tau} 1$, we have

$$(5.2) \quad A(\tau) = \sum_{\mu_k \geq (1+\tau)^{-1}} 1.$$

5.2. D is an n -dimensional rectangular parallelepiped and $g_{ij} = \delta_{ij}$.

Let D be an n -dimensional rectangular parallelepiped, $g_{ij} = \delta_{ij}$ in the whole of \bar{D} , $\varrho = \text{constant} > 0$ on one face Γ_ϱ^+ of the rectangular parallelepiped, i.e., $D = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i, i = 1, \dots, n\}$, $\Gamma_\varrho^+ = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = 0\}$ and $\Gamma_{00} = \Gamma^{l_n} = \{x \in \mathbb{R}^n | 0 \leq x_i \leq l_i \text{ when } i < n, x_n = l_n\}$. Without loss of generality, we assume $l_i < l_n$ for all $i < n$.

For the above domain D , except for the $K(D)$ and $K^d(D)$ in Section 3, we introduce the linear space of functions

$$K^0(D) = \{u | u \in H_0^1(D) \cap H^2(D) \cap C^\infty(\bar{D}), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_{00}, \Delta u = 0 \text{ on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma_{00})\}.$$

Clearly,

$$(5.3) \quad K^0(D) \subset K(D) \subset K^d(D).$$

Closing K^0 , K and K^d respect to the norm $\|u\| = \sqrt{\langle u, u \rangle}$, we obtain the Hilbert spaces \mathcal{K}^0 , \mathcal{K} and \mathcal{K}^d , and

$$(5.4) \quad \mathcal{K}^0 \subset \mathcal{K} \subset \mathcal{K}^d.$$

According to Theorem 3.3, we see that the bilinear functional

$$(5.5) \quad [u, v] = \int_{\Gamma_\varrho^+} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds$$

defines self-adjoint, completely continuous transformations G^0 , G and G^d on \mathcal{K}^0 , \mathcal{K} and \mathcal{K}^d , respectively (cf. Section 3). Obviously,

$$\langle G^0 u, v \rangle = \langle Gu, v \rangle \quad \text{for all } u, v \text{ in } \mathcal{K}^0,$$

$$\langle Gu, v \rangle = \langle G^d u, v \rangle \quad \text{for all } u, v \text{ in } \mathcal{K},$$

from which we deduce immediately by Proposition 2.3 that

$$(5.6) \quad \mu_k^0 \leq \mu_k \leq \mu_k^d, \quad k = 1, 2, 3, \dots,$$

where $\{\mu_k^0\}$ and $\{\mu_k^d\}$ are the eigenvalues of G^0 and G^d , respectively. Hence

$$(5.7) \quad A^0(\tau) \leq A(\tau) \leq A^d(\tau) \quad \text{for all } \tau,$$

where

$$(5.8) \quad A^0(\tau) = \sum_{\mu_k^0 \geq (1+\tau)^{-1}} 1$$

and

$$(5.9) \quad A^d(\tau) = \sum_{\mu_k^d \geq (1+\tau)^{-1}} 1.$$

We shall estimate the asymptotic behavior of $A^0(\tau)$ and $A^d(\tau)$. It is easy to verify (cf. Theorems 3.6, 3.7) that the eigenfunctions of the transformations G^0 and G^d , respectively, satisfy

$$(5.10) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma^{l_n}, \text{ and } \Delta u = 0 \text{ on } \Gamma - (\Gamma_\varrho \cup \Gamma^{l_n}), \\ \Delta u + \gamma \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

and

$$(5.11) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u = 0 & \text{on } \Gamma_\varrho^+, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}, \\ \frac{\partial(\Delta u)}{\partial \nu} = 0 & \text{and } \Delta u = 0 \text{ on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \Delta u + \kappa \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

As being verified in Section 4, the functions of form

$$(5.12) \quad u(x) = c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Y(x_n)$$

are the solutions of the problem (5.10), where m_1, \dots, m_{n-1} are positive integers, and $Y(x_n)$ is given by (4.8). Since the functions in (5.12) have derivatives of any order in D , it follows from Proposition 3.9 and Theorem 3.4 that they are eigenfunctions of the transformation G^0 with eigenvalues $(1 + \gamma)^{-1}$, where

$$(5.13) \quad \gamma = \frac{2\eta l_n}{\varrho l_n} \left(\frac{(\sinh \eta l_n) \cosh \eta l_n - \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right), \quad \eta = \left[\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2 \right]^{1/2}.$$

Note that the normal derivatives

$$(5.14) \quad \frac{\partial u}{\partial \nu} = c \left(\sin \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\sin \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right),$$

when m_1, \dots, m_{n-1} run through all positive integers (see, Section 4), form a complete system of orthogonal functions in $L_\varrho^2(\Gamma_\varrho^+)$. It follows from Proposition 3.5 that if m_1, \dots, m_{n-1}

run through all positive integers, then the functions (5.12) form an orthogonal basis of the subspace of \mathcal{K}^0 , spanned by the eigenfunctions of G^0 , corresponding to positive eigenvalues. That is, when m_1, \dots, m_{n-1} run through all positive integers, then $(1 + \gamma)^{-1}$, where γ is given by (5.13), runs through all positive eigenvalues of G^0 .

Similarly, for the problem (5.11), the eigenfunctions $\{u_k\}$ of the operator G^d on \mathcal{K}^d , corresponding to non-zero eigenvalues, form an orthogonal basis of the subspace of \mathcal{K}^d . The non-zero eigenvalues of G^d are $\mu_k^d = (1 + \kappa_k)^{-1}$, where κ_k is the k -th Steklov eigenvalue of (5.11).

In order to give the upper bound estimate of $A^d(\tau)$, we further introduce the following Steklov eigenvalue problem

$$(5.15) \quad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u = 0 \text{ on } \Gamma_\varrho^+, & u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ and } \frac{\partial(\Delta u)}{\partial \nu} = 0 & \text{on } \Gamma - (\Gamma_\varrho^+ \cup \Gamma^{l_n}), \\ \Delta u + \varsigma \varrho \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^+, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

Let ς_k be the k -th eigenvalue of (5.15). By Theorem 3.8, we have

$$(5.16) \quad \varsigma_k \leq \kappa_k \quad \text{for all } k \geq 1.$$

We define

$$(5.17) \quad \mu_k^f = \frac{1}{1 + \varsigma_k}, \quad A^f(\tau) = \sum_{\mu_k^f \geq (1+\tau)^{-1}} 1.$$

It follows from (5.16) and (5.17) that

$$(5.18) \quad A^d(\tau) \leq A^f(\tau) \quad \text{for all } \tau.$$

We know (cf. Section 4) that the problem (5.15) has the solutions of form

$$(5.19) \quad u(x) = c \left(\cos \frac{m_1 \pi}{l_1} x_1 \right) \cdots \left(\cos \frac{m_{n-1} \pi}{l_{n-1}} x_{n-1} \right) Z(x_n),$$

where m_1, \dots, m_{n-1} are non-negative integers with $\sum_{i=1}^{n-1} m_i \neq 0$, and $Z(x_n)$ is given by (4.13). This implies that if m_1, \dots, m_{n-1} run through all non-negative integers with $\sum_{i=1}^{n-1} m_i \neq 0$, then

$$(5.20) \quad \varsigma = \frac{2\beta l_n}{\rho l_n} \left(\frac{(\sinh \beta l_n) \cosh \beta l_n - \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right), \quad \beta = \left[\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l_i} \right)^2 \right]^{1/2}$$

runs throughout all eigenvalues of problem (5.15).

We first compute the asymptotic behavior of $A^f(\tau)$. By (5.17), (5.20) and the argument as in p. 44 of [50] or p. 373 of [6] or p. 51-53 of [39], $A^f(\tau)$ = the number of $(n-1)$ -tuples (m_1, \dots, m_{n-1}) satisfying the inequality

$$(5.21) \quad \frac{2\beta l_n}{\rho l_n} \left(\frac{(\sinh \beta l_n) \cosh \beta l_n - \beta l_n}{\sinh^2 \beta l_n - \beta^2 l_n^2} \right) \leq \tau,$$

where m_1, \dots, m_{n-1} are non-negative integers with $\sum_{i=1}^{n-1} m_i \neq 0$. By setting

$$(5.22) \quad t(s) = 2s \left(\frac{(\sinh s) \cosh s - s}{\sinh^2 s - s^2} \right),$$

we see that

$$\lim_{s \rightarrow +\infty} t(s)/s = 2.$$

We claim that for all $s \geq 1$,

$$t'(s) = 2 \left[\frac{-3s(\sinh^2 s) + 3s^2(\sinh s) \cosh s + (\sinh^3 s) \cosh s - s^3(\sinh^2 s + \cosh^2 s)}{(\sinh^2 s - s^2)^2} \right] > 0.$$

In fact, let

$$\theta(s) = -3s(\sinh^2 s) + 3s^3(\sinh s) \cosh s + (\sinh^3 s) \cosh s - s^3(\sinh^2 s + \cosh^2 s).$$

Then

$$\begin{aligned} \theta(1) &> 0, \text{ and} \\ \theta'(s) &= -3(\sinh^2 s) - 4s^3(\sinh s) \cosh s + 3(\sinh^2 s) \cosh^2 s + \sinh^4 s \\ &= 4(\sinh s)[\sinh^3 s - s^3 \cosh s] > 0 \quad \text{for } s \geq 1, \end{aligned}$$

This implies that $\theta(s) > 0$ for $s \geq 1$. Thus, the function $t(s)$ is increasing in $[1, +\infty)$. Denote by $s = h(t)$ the inverse of function $t(s)$ for $s \geq 1$. Then

$$\lim_{t \rightarrow +\infty} \frac{h(t)}{t} = \frac{1}{2}.$$

Furthermore, we can easily check that

$$(5.23) \quad h(t) \sim \frac{t}{2} + O(1) \quad \text{as } t \rightarrow +\infty.$$

Note that, for $s \geq 1$, the inequalities $t(s) \leq t$ is equivalent to $s \leq h(t)$. Hence (5.21) is equivalent to

$$\beta l_n \leq h(\varrho l_n \tau),$$

which can be written as

$$(5.24) \quad \sum_{i=1}^{n-1} (m_i/l_i)^2 \leq \left[\frac{1}{\pi l_n} h(\varrho l_n \tau) \right]^2, \quad m_i = 0, 1, 2, \dots$$

We consider the $(n-1)$ -dimensional ellipsoid

$$\sum_{i=1}^{n-1} (z_i/l_i)^2 \leq \left[\frac{1}{\pi l_n} h(\varrho l_n \tau) \right]^2.$$

Since $A^f(\tau) + 1$ just is the number of those $(n-1)$ -dimensional unit cubes of the z -space that have corners whose coordinates are non-negative integers in the ellipsoid (see, VI. §4 of [6]). Hence $A^f(\tau) + 1$ is the sum of the volumes of these cubes. Let $V(\tau)$ denote the volume and $T(\tau)$ the area of the part of the ellipsoid situated in the positive octant $z_i \geq 0, i = 1, \dots, n-1$. Then

$$(5.25) \quad V(\tau) \leq A^f(\tau) + 1 \leq V(\tau) + (n-1)^{\frac{1}{2}} T(\tau),$$

where $(n-1)^{\frac{1}{2}}$ is the diagonal length of the unit cube (see, [6] or [39]). Since

$$V(\tau) = \omega_{n-1} 2^{-(n-1)} l_1 \cdots l_{n-1} \left[\frac{h(\varrho l_n \tau)}{\pi l_n} \right]^{(n-1)},$$

by (5.23), we get that

$$(5.26) \quad V(\tau) \sim \omega_{n-1} (4\pi)^{-(n-1)} l_1 \cdots l_{n-1} \varrho^{n-1} \tau^{n-1} + O(\tau^{n-2}), \quad \text{as } \tau \rightarrow +\infty.$$

Note that

$$(5.27) \quad T(\tau) \sim \text{constant} \cdot \tau^{n-2}.$$

It follows that

$$\lim_{\tau \rightarrow +\infty} \frac{A^f(\tau)}{\tau^{n-1}} = \omega_{n-1} (4\pi)^{-(n-1)} l_1 \cdots l_{n-1} \varrho^{n-1},$$

i.e.,

$$(5.28) \quad A^f(\tau) \sim \frac{\omega_{n-1}}{(4\pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1} \tau^{n-1}, \quad \text{as } \tau \rightarrow +\infty,$$

where $|\Gamma_\varrho^+|$ denotes the area of the face Γ_ϱ^+ .

Next, we consider $A^0(\tau)$. Similarly,

$$(5.29) \quad \frac{2\eta l_n}{\varrho l_n} \left(\frac{(\sinh \eta l_n) \cosh \eta l_n - \eta l_n}{\sinh^2 \eta l_n - \eta^2 l_n^2} \right) \leq \tau,$$

is equivalent to

$$\eta l_n \leq h(\varrho l_n \tau),$$

i.e.,

$$\sum_{i=1}^{n-1} [m_i/l_i]^2 \leq \left(\frac{h(\varrho l_n \tau)}{\pi l_n} \right)^2, \quad m_i = 1, 2, 3, \dots$$

Similar to the argument for $A^f(\tau)$, we find (see also, [25] or §4 of [6]) that

$$\begin{aligned} \#\{(m_1, \dots, m_{n-1}) \mid \sum_{i=1}^{n-1} \left(\frac{m_i}{l_i} \right)^2 \leq \left(\frac{h(\varrho l_n \tau)}{\pi l_n} \right)^2, \quad m_i = 1, 2, 3, \dots\} \\ \sim \frac{\omega_{n-1}}{(4\pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty. \end{aligned}$$

i.e.,

$$(5.30) \quad \lim_{\tau \rightarrow +\infty} \frac{A^0(\tau)}{\tau^{n-1}} = \frac{\omega_{n-1}}{(4\pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1}.$$

Noting that $\varrho = 0$ on $\Gamma_\varrho - \Gamma_\varrho^+$, by (5.7), (5.18), (5.28) and (5.30), we have

$$(5.31) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(4\pi)^{(n-1)}} \int_{\Gamma_\varrho} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

5.3. A cylinder D whose base is an n -polyhedron of \mathbb{R}^{n-1} having $n-1$ orthogonal plane surfaces and $g_{ij} = \delta_{ij}$.

Lemma 5.1. *Let $D^{(r)} = \Gamma_\varrho^{(r)} \times [0, l_n]$, $r = 1, 2$, where $\Gamma_\varrho^{(1)} = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid x_i \geq 0 \text{ for } 1 \leq i \leq n-1, \text{ and } \sum_{i=1}^{n-1} \frac{x_i}{l_i} \leq 1\}$, and $\Gamma_\varrho^{(2)}$ is an $(n-1)$ -dimensional cube with side length $l = \max_{1 \leq i \leq n-1} l_i$. Assume that $\Gamma_{00}^{(r)} = \Gamma_\varrho^{(r)} \times \{l_n\}$, $r = 1, 2$. Assume also that ϱ is a positive constant on $\Gamma_\varrho^{(r)}$, $r = 1, 2$. If $l < l_n$, then*

$$(5.32) \quad \varsigma_k^f(D^{(1)}) \geq \varsigma_k^f(D^{(2)}) \quad \text{for } k = 1, 2, 3, \dots,$$

where $\varsigma_k^f(D^{(r)})$ (similar to ς of (3.30) in Theorem 3.8) is the k -th Steklov eigenvalue for the domain $D^{(r)}$.

Proof. Let $v_k^{(r)}$ be the k -th Neumann eigenfunction corresponding to $\alpha_k^{(r)}$ for the $(n-1)$ -dimensional domain $\Gamma_\varrho^{(r)}$, $(r = 1, 2)$, i.e.,

$$(5.33) \quad \begin{cases} \Delta v_k^{(r)} + \alpha_k^{(r)} v_k^{(r)} = 0 & \text{in } \Gamma_\varrho^{(r)}, \\ \frac{\partial v_k^{(r)}}{\partial \nu} = 0 & \text{on } \partial \Gamma_\varrho^{(r)}. \end{cases}$$

Put

$$u^{(r)}(x) = (v^{(r)}(x_1, \dots, x_{n-1}))(Z^{(r)}(x_n)) \quad \text{in } D^{(r)},$$

where $Z^{(r)}(x_n)$ is as in (4.13) with β being replaced by $\sqrt{\alpha_k^{(r)}}$. It is easy to verify that $u_k^{(r)}(x)$ satisfies

$$(5.34) \quad \begin{cases} \Delta^2 u_k^{(r)} = 0 & \text{in } D^{(r)}, \\ u_k^{(r)} = 0 & \text{on } \Gamma_\varrho^{(r)}, \\ u_k^{(r)} = \frac{\partial u_k^{(r)}}{\partial \nu} = 0 & \text{on } \Gamma_{00}^{(r)}, \\ \frac{\partial u_k^{(r)}}{\partial \nu} = \frac{\partial(\Delta u_k^{(r)})}{\partial \nu} = 0 & \text{on } \partial D^{(r)} - (\Gamma_\varrho^{(r)} \cup \Gamma_{00}^{(r)}), \\ \Delta u_k^{(r)} + \varsigma_k^f(D^{(r)}) \varrho \frac{\partial u_k^{(r)}}{\partial \nu} = 0 & \text{on } \Gamma_\varrho^{(r)}. \end{cases}$$

with

$$(5.35) \quad \varsigma_k^f(D^{(r)}) = \frac{2\sqrt{\alpha_k^{(r)}} l_n}{\varrho l_n} \left(\frac{(\sinh \sqrt{\alpha_k^{(r)}} l_n) \cosh \sqrt{\alpha_k^{(r)}} l_n - \sqrt{\alpha_k^{(r)}} l_n}{\sinh^2 \sqrt{\alpha_k^{(r)}} l_n - \alpha_k^{(r)} l_n^2} \right).$$

It follows from p. 437-438 of [6] that the k -th Neumann eigenvalue $\alpha_k^{(1)}$ for the domain $\Gamma_\varrho^{(1)}$ is at least as large as the k -th Neumann eigenvalue $\alpha_k^{(2)}$ for the domain $\Gamma_\varrho^{(2)}$. Recalling that $2s \left(\frac{(\sinh s) \cosh s - s}{\sinh^2 s - s^2} \right)$ is increasing when $s \geq 1$, we get

$$\varsigma_k^f(D^{(1)}) \geq \varsigma_k^f(D^{(2)}), \quad k = 1, 2, 3, \dots$$

if $l < l_n$. Here we have used the fact that $\sqrt{\alpha_k^{(2)}} l_n \geq 1$ since any Neumann eigenvalue for $\Gamma_\varrho^{(2)}$ has the form $\sum_{i=1}^{n-1} \left(\frac{m_i \pi}{l} \right)^2$. In other words, if $l < l_n$, then the number $A^f(\tau)$ of eigenvalues less than or equal to a given bound τ for the domain $D^{(1)}$ is at most equal to the corresponding number of eigenvalues for the domain $D^{(2)}$. \square

Similarly, we can easily verify that the number $A^f(\tau)$ of eigenvalues less than or equal to a given bound τ for an arbitrary n -dimensional rectangular parallelepiped D is never larger than the corresponding number for an n -dimensional rectangular parallelepiped of the same height with its base an $(n-1)$ -dimensional cube whose side length is at least equal to the largest side length of the base of D .

5.4. D is a cylinder and $g_{ij} = \delta_{ij}$.

Let D be an open n -dimensional cylinder in \mathbb{R}^n , whose boundary consists of an $(n-1)$ -dimensional cylindrical surface and two parallel plane surfaces perpendicular to the cylindrical surface. Assume that $g_{ij} = \delta_{ij}$ in the whole of \bar{D} , that Γ_ϱ includes at least one of the plane surfaces, which we call Γ_ϱ^+ , and that ϱ is positive constant on Γ_ϱ^+ and vanishes on $\Gamma_\varrho - \Gamma_\varrho^+$. We let the plane surface Γ_ϱ^+ be situated in the plane $x_n = 0$ and let another parallel surface Γ^{l_n} be situated in the plane $\{x \in \mathbb{R}^n | x_n = l_n\}$. We now divide the plane $x_n = 0$ into a net of $(n-1)$ -dimensional cubes, whose faces are parallel to the

coordinate-planes in $x_n = 0$. Let $\Gamma_1, \dots, \Gamma_p$ be those open cubes in the net, closure of which are entirely contained in Γ_ϱ^+ , and let Q_{p+1}, \dots, Q_q be the remaining open cubes, whose closure intersect Γ_ϱ^+ . We may let the subdivision into cubes be so fine that, for every piece of the boundary of Γ_ϱ^+ which is contained in one of the closure cubes, the direction of the normal varies by less than a given angle ϑ , whose size will be determined later. (This can be accomplished by repeated halving of the side of cube.) We can make the side length l of each cube be less than l_n . Furthermore, let D_j , ($j = 1, \dots, p$), be the open n -dimensional rectangular parallelepiped with the cube Γ_j as a base and otherwise bounded by the “upper” plane surface $\Gamma_j^{l_n}$ of the cylinder \bar{D} and planes parallel to the coordinate-planes $x_1 = 0, \dots, x_{n-1} = 0$ (cf. [39]).

We define the linear spaces of functions

$$\begin{aligned} K &= \{u \mid u \in H_0^1(D) \cap H^2(D), \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_{00}\}, \\ K_j^0 &= \{u_j \mid u_j \in H_0^1(D_j) \cap H^2(D_j) \cap C^\infty(\bar{D}_j), \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^{l_n}, \text{ and} \\ &\quad \Delta u_j = 0 \text{ on } \partial D_j - (\Gamma_j \cup \Gamma_j^{l_n})\}, \quad (j = 1, \dots, p) \end{aligned}$$

with the inner products

$$\begin{aligned} \langle u, v \rangle &= \int_D (\Delta u)(\Delta v) dR + \int_{\Gamma_\varrho} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds \quad \text{for } u, v \in K, \\ \langle u_j, v_j \rangle_j &= \langle u_j, v_j \rangle_j^* + [u_j, v_j]_j = \int_{D_j} (\Delta u_j)(\Delta v_j) dR + \int_{\Gamma_j} \varrho \frac{\partial u_j}{\partial \nu} \frac{\partial v_j}{\partial \nu} ds \quad \text{for } u_j, v_j \in K_j^0, \end{aligned}$$

respectively. Closing K and K_j^0 with respect to the norms $\|u\| = \sqrt{\langle u, u \rangle}$ and $\|u_j\|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we obtain the Hilbert spaces \mathcal{K} and \mathcal{K}_j^0 ($j = 1, \dots, p$), respectively. Clearly, the bilinear functional

$$\begin{aligned} [u, v] &= \int_{\Gamma_\varrho} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds \\ [u_j, v_j]_j &= \int_{\Gamma_j} \varrho \frac{\partial u_j}{\partial \nu} \frac{\partial v_j}{\partial \nu} ds, \quad (j = 1, \dots, p), \end{aligned}$$

define self-adjoint, completely continuous transformations G and G_j^0 on \mathcal{K} and \mathcal{K}_j^0 by

$$(5.36) \quad \langle Gu, v \rangle = [u, v] \quad \text{for } u, v \text{ in } \mathcal{K},$$

$$(5.37) \quad \langle G_j^0 u_j, v_j \rangle_j = [u_j, v_j]_j \quad \text{for } u_j \text{ and } v_j \text{ in } \mathcal{K}_j^0,$$

respectively. By defining a space

$$\mathcal{K}^0 = \sum_{j=1}^p \oplus \mathcal{K}_j^0 = \{u^0 \mid u^0 = u_1 + \dots + u_p, \ u_j \in \mathcal{K}_j^0\}$$

with its inner product

$$(5.38) \quad \langle u^0, v^0 \rangle = \sum_{j=1}^p \langle u_j, v_j \rangle_j,$$

we find that the space \mathcal{K}^0 becomes a Hilbert space. If we define the transformation G^0 on \mathcal{K}^0 by

$$(5.39) \quad G^0 u^0 = G_1^0 u_1 + \dots + G_p^0 u_p \quad \text{for } u^0 = u_1 + \dots + u_p \text{ in } \mathcal{K}^0,$$

we see that G^0 is a self-adjoint, completely continuous transformation on \mathcal{K}^0 . If we put

$$(5.40) \quad [u^0, v^0] = \sum_{j=1}^p [u_j, v_j]_j,$$

we find by (5.37)—(5.40) that

$$(5.41) \quad \langle G^0 u^0, v^0 \rangle = [u^0, v^0] \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

Let us define a mapping of \mathcal{K}^0 into \mathcal{K} . Let $u^0 = u_1 + \cdots + u_p$, $u_j \in H_j^0$, be an element of \mathcal{K}^0 and define

$$(5.42) \quad u = \Pi^0 u^0,$$

where $u(x) = u_j(x)$, when $x \in \bar{D}_j$, and $u(x) = 0$, when $x \in \bar{D} - \cup_{j=1}^p \bar{D}_j$. Clearly $u \in \mathcal{K}$ and thus (5.42) defines a transformation Π^0 of $\mathcal{K}_1^0 \oplus \cdots \oplus \mathcal{K}_p^0$ into \mathcal{K} . It is readily seen that

$$(5.43) \quad [\Pi^0 u^0, \Pi^0 v^0] = [u^0, v^0] \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

and

$$(5.44) \quad \langle G^0 u^0, v^0 \rangle = \langle G \Pi^0 u^0, \Pi^0 v^0 \rangle \quad \text{for all } u^0 \text{ and } v^0 \text{ in } \mathcal{K}^0.$$

By (5.43) and (5.44), we find by applying Proposition 2.3 that

$$\mu_k^0 \leq \mu_k \quad \text{for } k = 1, 2, 3, \dots.$$

Therefore

$$(5.45) \quad A^0(\tau) \leq A(\tau).$$

The definition of G^0 implies that

$$(5.46) \quad G^0 \mathcal{K}_j^0 \subset \mathcal{K}_j^0, \quad (j = 1, \dots, p),$$

and

$$(5.47) \quad G^0 u^0 = G_j^0 u^0, \quad \text{when } u^0 \in \mathcal{K}_j^0.$$

From (5.40), (5.41), (5.46), (5.47) and Proposition 2.4, we obtain

$$(5.48) \quad A^0(\tau) = \sum_{j=1}^p A_j^0(\tau),$$

where $A_j^0(\tau)$ is the number of eigenvalues of the transformation G_j^0 on \mathcal{K}_j^0 which are greater or equal to $(1 + \tau)^{-1}$. Because \bar{D}_j , $(j = 1, \dots, p)$, is an n -dimensional rectangular parallelepiped we find by (5.30) that

$$(5.49) \quad A_j^0(\tau) \sim \omega_{n-1} (4\pi)^{-(n-1)} |\Gamma_j| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty,$$

where $|\Gamma_j|$ denotes the area of the face Γ_j of D_j . By (5.48) and (5.49) we infer that

$$(5.50) \quad A^0(\tau) \sim \omega_{n-1} (4\pi)^{-(n-1)} \sum_{j=1}^p |\Gamma_j| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty.$$

Next, we shall calculate the upper estimate of $A(\tau)$. Let \bar{P}_j , $(j = p+1, \dots, q)$, be the n -dimensional rectangular parallelepiped with the cube \bar{Q}_j as a base and otherwise bounded by the “upper” plane surface Γ^{l_n} of the cylinder \bar{D} and planes parallel to the coordinate-planes $x_1 = 0, \dots, x_{n-1} = 0$. The intersection $\bar{P}_j \cap \bar{D}$ is a cylinder \bar{D}_j , $(j = p+1, \dots, q)$, with $\bar{\Gamma}_j := \bar{Q}_j \cap \bar{\Gamma}_\varrho^+$ as a base. Obviously

$$(5.51) \quad \bar{D} = \sum_{j=1}^q \bar{D}_j.$$

We first define the linear spaces of functions

$$K^d = \{u | u \in H^2(D), u = 0 \text{ on } \Gamma_\varrho, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^{l_n}\},$$

$$K_j^d = \{u_j | u_j \in H^2(D_j), u_j = 0 \text{ on } \Gamma_j, u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^{l_n}\}, \quad (j = 1, \dots, q)$$

with the inner products

$$(5.52) \quad \langle u, v \rangle = \int_D (\Delta u)(\Delta v) dR + \int_{\Gamma_\varrho} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds,$$

and

$$(5.53) \quad \langle u_j, v_j \rangle_j = \int_{D_j} (\Delta u_j)(\Delta v_j) dR + \int_{\Gamma_j} \varrho \frac{\partial u_j}{\partial \nu} \frac{\partial v_j}{\partial \nu} ds,$$

respectively. Closing K^d and K_j^d with respect to the norms $\|u\| = \sqrt{\langle u, u \rangle}$ and $\|u_j\|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we obtain Hilbert spaces \mathcal{K}^d and \mathcal{K}_j^d , ($j = 1, \dots, q$), and then we define the Hilbert space

$$(5.54) \quad \mathcal{K}^d = \sum_{j=1}^q \oplus \mathcal{K}_j^d = \{u^d | u^d = u_1 + \dots + u_q, u_j \in \mathcal{K}_j^d\}$$

with its inner product

$$(5.55) \quad \langle u^d, v^d \rangle = \sum_{j=1}^q \langle u_j, v_j \rangle_j.$$

The bilinear functional

$$(5.56) \quad [u_j, v_j]_j = \int_{\Gamma_j} \varrho \frac{\partial u_j}{\partial \nu} \frac{\partial v_j}{\partial \nu} ds, \quad (j = 1, \dots, q),$$

define a self-adjoint, completely continuous transformation G_j^d on \mathcal{K}_j^d given by

$$(5.57) \quad \langle G_j^d u_j, v_j \rangle_j = [u_j, v_j]_j \quad \text{for all } u_j \text{ and } v_j \text{ in } \mathcal{K}_j^d.$$

The self-adjoint, completely continuous transformation G^d on \mathcal{K}^d is defined by

$$(5.58) \quad G^d u^d = G_1^d u_1 + \dots + G_q^d u_q \quad \text{for } u^d = u_1 + \dots + u_q \text{ in } \mathcal{K}^d.$$

With

$$(5.59) \quad [u^d, v^d] = \sum_{j=1}^q [u_j, v_j]_j,$$

we find by (5.55), (5.57)—(5.59) that

$$(5.60) \quad \langle G^d u^d, v^d \rangle = [u^d, v^d] \quad \text{for all } u^d \text{ and } v^d \text{ in } \mathcal{K}^d.$$

Let us define a mapping Π of \mathcal{K} into \mathcal{K}^d . Let $u \in K(D)$, and put

$$u^d = \Pi u = u_1 + \dots + u_q,$$

where $u_j(x) = u(x)$, when $x \in \bar{D}_j$. It can be easily verified that

$$(5.61) \quad \langle \Pi u, \Pi v \rangle = \langle u, v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}.$$

and

$$(5.62) \quad \langle Gu, v \rangle = \langle G^d \Pi u, \Pi v \rangle \quad \text{for all } u \text{ and } v \text{ in } \mathcal{K}.$$

From (5.61)—(5.62), with the aid of Proposition 2.3, we obtain

$$\mu_k \leq \mu_k^d \quad \text{for } k = 1, 2, 3, \dots,$$

and hence

$$(5.63) \quad A(\tau) \leq A^d(\tau).$$

By $G^d \mathcal{K}_j^d \subset \mathcal{K}_j^d$, ($j = 1, \dots, q$), and $G^d u^d = G_j^d u^d$ when $u^d \in \mathcal{K}_j^d$, we get

$$(5.64) \quad A^d(\tau) = \sum_{j=1}^q A_j^d(\tau),$$

where $A_j^d(\tau)$ is the number of eigenvalues of the transformation G_j^d on \mathcal{K}_j^d which are greater than or equal to $(1 + \tau)^{-1}$. Further, we define $A_j^f(\tau)$ similar to (5.15) and (5.17), i.e.,

$$A_j^f(\tau) = \sum_{\mu_k^f \geq (1+\tau)^{-1}} 1 \quad \text{with} \quad \mu_k^f = \frac{1}{1 + \varsigma_k},$$

where ς_k is the k -th Steklov eigenvalue of the following problem

$$\begin{cases} \Delta^2 u_j = 0 & \text{in } D_j, \\ u_j = 0 \text{ on } \Gamma_j, & u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^{l_n}, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u_j)}{\partial \nu} = 0 & \text{on } \partial D_j - (\Gamma_j \cup \Gamma_j^{l_n}), \\ \Delta u_j + \varsigma \varrho \frac{\partial u_j}{\partial \nu} = 0 & \text{on } \Gamma_j, \quad \varrho = \text{constant} > 0 \text{ on } \Gamma_\varrho^+. \end{cases}$$

From Theorem 3.8, it follows that

$$\varsigma_k \leq \kappa_k \quad \text{for all } k \geq 1,$$

and hence

$$(5.65) \quad A_j^d(\tau) \leq A_j^f(\tau) \quad \text{for all } \tau \text{ and } j = 1, \dots, q,$$

where $\frac{1}{1+\kappa_k}$ is the k -th eigenvalue of the transformation G_j^d . Since \bar{D}_j , ($j = 1, \dots, p$), is an n -dimensional rectangular parallelepiped, we find from (5.28) that

$$(5.66) \quad A_j^f(\tau) \sim \omega_{n-1} (4\pi)^{-(n-1)} |\Gamma_j| \varrho^{n-1} \tau^{n-1}, \quad (j = 1, \dots, p).$$

It remains to estimate $A_j^f(\tau)$, ($j \geq p+1$). According to the argument in p.438-440 of [6], each of the $(n-1)$ -dimensional domains Γ_j is bounded either by $n-1$ orthogonal plane surfaces of the partition (the diameter of the intersection of any two plane surfaces lies between l and $3l$), and an $(n-2)$ -dimensional surface of the boundary (see, in two dimensional case, Figure 5 of p.439 of [6]), or by $2n-3$ orthogonal plane surfaces of the partition (the diameter of the intersection of any two plane surfaces lies between l and $3l$), and a surface of the boundary $\partial\Gamma_\varrho$ (see, in two dimensional case, Figure 6 of p.439 of [6]). The number $q-p$ is evidently smaller than a constant C/l^{n-2} , where C is independent of l and depends essentially on the area of the boundary $\partial\Gamma_\varrho$. Now, we take any point on the boundary surface of Γ_j and take the tangent plane through it. This tangent plane together with the plane parts of $\partial\Gamma_j$ bounds an n -polyhedron of \mathbb{R}^{n-1} with a vertex at which $n-1$ orthogonal plane surfaces meet (see, Figure 5 of p.439 of [6] in two dimensions), e.g., if ϑ is sufficiently small it forms an $(n-1)$ -dimensional n -polyhedron of \mathbb{R}^n with a vertex having $n-1$ orthogonal plane surfaces (the diameter of the intersection of any two plane surfaces is also smaller than $4l$), or else an $(n-1)$ -dimensional $2(n-1)$ -polyhedron of \mathbb{R}^{n-1} (see, Figure 6 of p.439 of [6] in two dimensional case), the diameter of the intersection of any two plane surfaces (except for the top inclined plane surface) of the $2(n-1)$ -polyhedron is also smaller than $4l$; The shape of the result domain depends on the type to which $\bar{\Gamma}_j$ belongs. We shall denote the result domains by S'_j . The domain Γ_j can always be deformed into the domain S'_j by a transformation of the form (2.1), as defined in Section 2. In the case

of domains of the first type, let the intersection point of $n - 1$ orthogonal plane surfaces be the pole of a system of pole coordinates $r, \theta_1, \theta_2, \dots, \theta_{n-2}$, and let $r = f(\theta_1, \theta_2, \dots, \theta_{n-2})$ be the equation of the boundary surface of Γ_ρ , $r = h(\theta_1, \theta_2, \dots, \theta_{n-2})$ the equation of the inclined plane surface of the n -polyhedron of \mathbb{R}^{n-1} having a vertex of $n - 1$ orthogonal plane surfaces. Then the equations

$$\theta'_1 = \theta_1, \quad \theta'_2 = \theta_2, \quad \dots, \quad \theta'_{n-2} = \theta_{n-2}, \quad r' = r \frac{h(\theta_1, \theta_2, \dots, \theta_{n-2})}{f(\theta_1, \theta_2, \dots, \theta_{n-2})}$$

represents a transformation of the domain Γ_j into the n -polyhedron S'_j of \mathbb{R}^{n-1} . For a domain of the second type, let $x_{n-1} = h(x_1, \dots, x_{n-2})$ be the equation of top plane surface of the $2(n - 1)$ -polyhedron and let $x_{n-1} = f(x_1, \dots, x_{n-2})$ be the equation of the boundary surface of Γ_ρ . We then consider the transformation

$$x'_1 = x_1, \quad \dots, \quad x'_{n-2} = x_{n-2}, \quad x'_{n-1} = x_{n-1} \frac{h(x_1, \dots, x_{n-2})}{f(x_1, \dots, x_{n-2})}.$$

If we assume that the side length l of cube in the partition is sufficiently small, and therefore the rotation of the normal on the boundary surface is taken sufficiently small, then the transformations considered here evidently have precise the form (2.1), and the quantity denoted by ϵ in (2.1) is arbitrarily small. From Corollary to Theorem 10 of p. 423 of [6], we know that there exists a number $\delta > 0$ depending on ϵ and approaching zero with ϵ , such that

$$\left| \frac{\alpha_k(S'_j)}{\alpha_k(\Gamma_j)} - 1 \right| < \delta \quad \text{uniformly for all } k,$$

where $\alpha_k(\Gamma_j)$ and $\alpha_k(S'_j)$ are the k -th Neumann eigenvalues of Γ_j and S'_j , respectively. According to the argument as in the proof of Lemma 5.1 (i.e., (5.35)), we see that

$$\varsigma_k^f(E_j) = \frac{1}{\varrho l_n} t(l_n \sqrt{\alpha_k(\Gamma_j)}), \quad \varsigma_k^f(E'_j) = \frac{1}{\varrho l_n} t(l_n \sqrt{\alpha_k(S'_j)}),$$

where $t(s)$ is given by (5.22), and $\varsigma_k^f(E_j)$ and $\varsigma_k^f(E'_j)$ (similar to ς of (3.30)) are the k -th Steklov eigenvalue for the n -dimensional domains $E_j = \Gamma_j \times [0, l_n]$ and $E'_j = S'_j \times [0, l_n]$, respectively. Recalling that the function $t = t(s)$ is continuous and increasing for $s \geq 1$, we get that there exists a constant $\delta' > 0$ depending on ϵ approaching zero with ϵ , such that

$$\left| \frac{\varsigma_k^f(E'_j)}{\varsigma_k^f(E_j)} - 1 \right| < \delta'.$$

In other words, the corresponding k -th Steklov eigenvalues for the n -dimensional domains $E_j = \Gamma_j \times [0, l_n]$ and $E'_j = S'_j \times [0, l_n]$ differ only by a factor which itself differs by a small amount from 1, uniformly for all k . Therefore, the same is true also for the corresponding numbers $A_{E_j}^f(\tau)$ and $A_{E'_j}^f(\tau)$ of the eigenvalues less or equal to the bound τ .

The domain E'_j is either a cylinder whose base is an n -polyhedron of \mathbb{R}^{n-1} having $(n - 1)$ orthogonal plane surfaces with its largest side length smaller than $4l$ or a cylinder whose base is a combination of such an n -polyhedron of \mathbb{R}^{n-1} and an $(n - 1)$ -dimensional cube with sides smaller than $3l$; it follows from the estimates for E'_j (cf. (5.25)–(5.27)) and Lemma 5.1 that if l is taken sufficiently small, the number $A_{E'_j}^f(\tau)$ from some τ on satisfies the inequality

$$A_{E'_j}^f(\tau) < C_1 l^{n-1} \tau^{n-1} + C_2 l^{n-2} \tau^{n-2}$$

where C_1, C_2 are constants, to be chosen suitably. Thus, $A_{E_j}^f(\tau)$ can be written as $A_{E_j}^f(\tau) = \theta(C_3 l^{n-1} \tau^{n-1} + C_4 l^{n-2} \tau^{n-2})$, where θ denotes a number between -1 and $+1$ and C_3, C_4 are constants independent of l, j and τ . It follows that

$$\sum_{j=p+1}^q A_{E_j}^f(\tau) = \tau^{n-1} [\theta C_3 (q-p) l^{n-1} + \theta C_4 (q-p) l^{n-2} \frac{1}{\tau}].$$

As pointed out before, $(q-p)l^{n-2} < C$; therefore, for sufficiently small l , $(q-p)l^{n-1}$ is arbitrarily small and we have the asymptotic relation

$$(5.67) \quad \lim_{\tau \rightarrow +\infty} \sum_{j=p+1}^q \frac{A_{E_j}^f(\tau)}{\tau^{n-1}} = \varpi(l),$$

where $\varpi(l) \rightarrow 0$ as $l \rightarrow 0$. For, we may choose the quantity l arbitrarily, and by taking a sufficiently small fixed l , make the factors of τ^{n-1} in the previous equalities arbitrarily close to zero for sufficiently large τ . Since

$$(5.68) \quad A_{E_j}^d(\tau) \leq A_{E_j}^f(\tau) \quad \text{for } j = p+1, \dots, q,$$

we get

$$(5.69) \quad \lim_{\tau \rightarrow +\infty} \frac{\sum_{j=p+1}^q A_{E_j}^d(\tau)}{\tau^{n-1}} \leq \lim_{\tau \rightarrow +\infty} \frac{\sum_{j=p+1}^q A_{E_j}^f(\tau)}{\tau^{n-1}} = \varpi(l).$$

From (5.45), (5.50), (5.63), (5.65), (5.66), (5.67), (5.68) and (5.69), we obtain

$$(5.70) \quad \begin{aligned} \omega_{n-1} (4\pi)^{-(n-1)} \varrho^{n-1} \sum_{j=1}^p |\Gamma_j| &\leq \lim_{\tau \rightarrow \infty} \frac{A(\tau)}{\tau^{n-1}} \leq \overline{\lim}_{\tau \rightarrow \infty} \frac{A(\tau)}{\tau^{n-1}} \\ &\leq \left(\omega_{n-1} (4\pi)^{-(n-1)} \varrho^{n-1} \sum_{j=1}^p |\Gamma_j| \right) + \varpi(l). \end{aligned}$$

Letting $l \rightarrow 0$, we immediately see that $\sum_{j=1}^p |\Gamma_j|$ tends to the area $|\Gamma_\varrho|$ of Γ_ϱ and $\lim_{l \rightarrow 0} \varpi(l) = 0$. Therefore, (5.70) gives

$$(5.71) \quad A(\tau) \sim \frac{\omega_{n-1}}{(4\pi)^{(n-1)}} |\Gamma_\varrho^+| \varrho^{n-1} \tau^{n-1} \quad \text{as } \tau \rightarrow +\infty,$$

or

$$(5.72) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(4\pi)^{(n-1)}} \int_{\Gamma_\varrho} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

Remark 5.2. In the above argument, we first made the assumption that the boundary $\partial\Gamma_\varrho$ of Γ_ϱ was smooth. However, the corresponding discussion and result remain essentially valid if $\partial\Gamma_\varrho$ is composed of a finite number of $(n-2)$ dimensional smooth surfaces.

6. PROOFS OF MAIN RESULTS

Lemma 6.1. *Let g_{ij} and g'_{ij} be two metric tensors on manifold \mathcal{M} such that*

$$(6.1) \quad |g^{ij} - g'^{ij}| < \epsilon, \quad i, j = 1, \dots, n$$

and

$$(6.2) \quad \left| \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{ij}) - \frac{1}{\sqrt{|g'|}} \frac{\partial}{\partial x_i} (\sqrt{|g'|} g'^{ij}) \right| \leq \epsilon, \quad i, j = 1, \dots, n$$

for all points in \bar{D} , where D is a bounded domain in \mathcal{M} . Let

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq \dots > 0 \quad \text{and} \quad \mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_n \geq \dots > 0$$

be positive eigenvalues of G and G' , respectively, where G and G' are given by

$$\begin{aligned} \langle Gu, v \rangle &= \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds, \quad \text{for } u \text{ and } v \text{ in } \mathcal{K}, \\ \langle G'u, v \rangle' &= \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds', \quad \text{for } u \text{ and } v \text{ in } \mathcal{K}'. \end{aligned}$$

Then

$$(6.3) \quad \begin{aligned} (1 + \tilde{M}\epsilon)^{-(n+1)/2} \left(\max\{(1 + \epsilon M), (1 + \tilde{M}\epsilon)^{(n+1)/2}\} \right)^{-1} \mu_k &\leq \mu'_k \\ &\leq (1 + \tilde{M}\epsilon)^{(n+1)/2} \left(\min\{(1 - \epsilon M), (1 + \tilde{M}\epsilon)^{-(n+1)/2}\} \right)^{-1} \mu_k, \\ &\text{for } k = 1, 2, 3, \dots, \end{aligned}$$

where \tilde{M} and M are constants depending only on $g, g', \frac{\partial g_{ij}}{\partial x_l}, \frac{\partial g'^{ij}}{\partial x_l}, \frac{\partial g'_{ij}}{\partial x_l}, \frac{\partial g'^{ij}}{\partial x_l}$ and \bar{D} .

Proof. It follows from (6.1) that there exists a positive constant \tilde{M} independent of ϵ and depending only on g^{ij}, g'^{ij} and \bar{D} such that

$$(1 + \epsilon \tilde{M})^{-1} \sum_{i,j=1}^n g^{ij} t_i t_j \leq \sum_{i,j=1}^n g'^{ij} t_i t_j \leq (1 + \epsilon \tilde{M}) \sum_{i,j=1}^n g^{ij} t_i t_j$$

for all points in \bar{D} and all real numbers t_1, \dots, t_n . Thus we have

$$(1 + \tilde{M}\epsilon)^{-n/2} \sqrt{|g|} \leq \sqrt{|g'|} \leq (1 + \tilde{M}\epsilon)^{n/2} \sqrt{|g|},$$

which implies (see p. 64-65 of [39]) that

$$(1 + \tilde{M}\epsilon)^{-n/2} dR \leq dR' \leq (1 + \tilde{M}\epsilon)^{n/2} dR$$

and

$$(1 + \tilde{M}\epsilon)^{-(n+1)/2} ds \leq ds' \leq (1 + \tilde{M}\epsilon)^{(n+1)/2} ds.$$

Thus

$$(6.4) \quad (1 + \tilde{M}\epsilon)^{-(n+1)/2} [u, u] \leq [u, u]' \leq (1 + \tilde{M}\epsilon)^{(n+1)/2} [u, u].$$

Putting

$$\omega_{ij} = g'^{ij} - g^{ij}, \quad \theta_{ij} = \frac{1}{\sqrt{|g'|}} \frac{\partial}{\partial x_i} (\sqrt{|g'|} g'^{ij}) - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{ij}),$$

we immediately see that

$$\max_{x \in \bar{D}} |\omega_{ij}| \leq \epsilon \quad \text{and} \quad \max_{x \in \bar{D}} |\theta_{ij}| \leq \epsilon.$$

Thus, for any $u \in \mathcal{K}^0(D)$ or $u \in \mathcal{K}^d(D)$, we have

$$\Delta_{g'} u = \sum_{i,j=1}^n (\omega_{ij} + g^{ij}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^n \left[\theta_{ij} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{ij}) \right] \frac{\partial u}{\partial x_j},$$

so that

$$\Delta_{g'}u - \Delta_g u = \sum_{i,j=1}^n \left[\omega_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \theta_{ij} \frac{\partial u}{\partial x_j} \right]$$

It follows that

$$|\Delta_{g'}u - \Delta_g u| \leq \epsilon(M_1 |\nabla_g^2 u| + M_2 |\nabla_g u|),$$

where $|\nabla_g^2 u|^2$ is defined in an invariant ways as

$$|\nabla_g^2 u|^2 = \nabla^l \nabla^k u \nabla_l \nabla_k u = g^{pl} g^{kq} \left(\frac{\partial^2 u}{\partial x_k \partial x_l} - \Gamma_{kl}^m \frac{\partial u}{\partial x_m} \right) \left(\frac{\partial^2 u}{\partial x_p \partial x_q} - \Gamma_{pq}^r \frac{\partial u}{\partial x_r} \right),$$

and M_1 and M_2 are constants depending only on $g, g', \frac{\partial g_{ij}}{\partial x_l}, \frac{\partial g^{ij}}{\partial x_l}, \frac{\partial g'_{ij}}{\partial x_l}, \frac{\partial g'^{ij}}{\partial x_l}$ and \bar{D} . Thus,

$$(6.5) \quad \int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(M_1^2 \int_D |\nabla_g^2 u|^2 dR + M_2^2 \int_D |\nabla_g u|^2 dR \right).$$

Recall also (see, Section 2) that $\Lambda_1^0(D) \geq \Lambda_1^d(D)$, where

$$(6.6) \quad \Lambda_1^0(D) = \inf_{v \in K^0(D), \int_D |\nabla_g v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g v|^2 dR},$$

$$(6.7) \quad \Lambda_1^d(D) = \inf_{v \in K^d(D), \int_D |\nabla_g v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g v|^2 dR},$$

and $K^0(D)$ and $K^d(D)$ are as in Section 3. Let

$$(6.8) \quad \Theta_1^0(D) = \inf_{v \in K^0(D), \int_D |\nabla_g^2 v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g^2 v|^2 dR},$$

$$(6.9) \quad \Theta_1^d(D) = \inf_{v \in K^d(D), \int_D |\nabla_g^2 v|^2 dR = 1} \frac{\int_D |\Delta_g v|^2 dR}{\int_D |\nabla_g^2 v|^2 dR}.$$

Clearly, $\Theta_1^0(D) \geq \Theta_1^d(D)$. As in the proofs of Lemmas 2.1, 2.2, it is easy to prove that the existence of the minimizers to (6.7) and (6.9), respectively. Therefore, we have that $\Lambda_1^d(D) > 0$ and $\Theta_1^d(D) > 0$ (Suppose by contradiction that $\Lambda_1^d(D) = 0$ and $\Theta_1^d(D) = 0$. Then $\Delta_g u = 0$ in D for the corresponding minimizer $u \in K^d(D)$ in both cases. By applying Holmgren's uniqueness theorem for the minimizer $u \in K^d(D)$ in both cases, we immediately see that $u \equiv 0$ in D . This contradicts the assumption $\int_D |\nabla_g u|^2 dR = 1$ or $\int_D |\nabla_g^2 u|^2 dR = 1$ for the minimizer $u \in K^d(D)$ in the corresponding cases). Combining these inequalities, we obtain

$$\int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(\frac{M_1^2}{\Theta_1^0(D)} + \frac{M_2^2}{\Lambda_1^0(D)} \right) \int_D |\Delta_g u|^2, \quad \text{for } u \in K^0(D)$$

and

$$\int_D |\Delta_{g'}u - \Delta_g u|^2 dR \leq 2\epsilon^2 \left(\frac{M_1^2}{\Theta_1^d(D)} + \frac{M_2^2}{\Lambda_1^d(D)} \right) \int_D |\Delta_g u|^2 \quad \text{for } u \in K^d(D).$$

Thus we have that, for all $u \in K^0(D)$ or $u \in K^d(D)$,

$$(1 - \epsilon M) \int_D |\Delta_g u|^2 dR \leq \int_D |\Delta_{g'}u|^2 dR \leq (1 + \epsilon M) \int_D |\Delta_g u|^2 dR,$$

where M is a constant depending only $g, g', \frac{\partial g_{ij}}{\partial x_l}, \frac{\partial q^{ij}}{\partial x_l}, \frac{\partial g'_{ij}}{\partial x_l}, \frac{\partial q'^{ij}}{\partial x_l}$ and \bar{D} . That is,

$$(6.10) \quad (1 - \epsilon M) \langle u, u \rangle^* \leq \langle u, u \rangle'^* \leq (1 + \epsilon M) \langle u, u \rangle^*.$$

By (6.4) and (6.10) we obtain that, for all $u \in K^0(D)$ or $u \in K^d(D)$,

$$\begin{aligned} & \frac{(1 + \tilde{M}\epsilon)^{-(n+1)/2} [u, u]}{(\max\{(1 + \epsilon M), (1 + \tilde{M}\epsilon)^{(n+1)/2}\}) (\langle u, u \rangle^* + [u, u])} \leq \frac{[u, u]'}{\langle u, u \rangle'^* + [u, u]'} \\ & \leq \frac{(1 + \tilde{M}\epsilon)^{(n+1)/2} [u, u]}{(\min\{(1 - \epsilon M), (1 + \tilde{M}\epsilon)^{-(n+1)/2}\}) (\langle u, u \rangle^* + [u, u])}, \end{aligned}$$

which implies (6.3). \square

Remark 6.2. Let $\tilde{\Gamma}$ and Γ be two bounded domains in \mathbb{R}^{n-1} , let $\tilde{\Gamma}$ is similar to Γ (in the elementary sense of the term; the length of any line in $\tilde{\Gamma}$ is to the corresponding length in Γ as h to 1), and let $\Gamma_{00} = \Gamma \times \{\sigma\}$ and $\tilde{\Gamma}_{00} = \tilde{\Gamma} \times \{h\sigma\}$. It is easy to verify that

$$\Lambda_1^d(\tilde{D}) = h^{-2} \Lambda_1^d(D), \quad \Theta_1^d(\tilde{D}) = \Theta_1^d(D),$$

where $D = \Gamma_i \times [0, \sigma]$, $\tilde{D} = \tilde{\Gamma} \times [0, h\sigma]$, and $\Lambda_1^d(D)$ and $\Theta_1^d(D)$ are defined as in (6.7) and (6.9), respectively.

Lemma 6.3. Let G and G' be the continuous linear transformations defined by

$$\langle Gu, v \rangle = \int_{\Gamma_e} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds \quad \text{for } u \text{ and } v \text{ in } K^0(D) \text{ or } K^d(D)$$

and

$$\langle G'u, v \rangle' = \int_{\Gamma_e} \varrho' \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds \quad \text{for } u \text{ and } v \text{ in } K^0(D) \text{ or } K^d(D),$$

respectively. Let

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots > 0 \quad \text{and} \quad \mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_k \geq \cdots > 0$$

be the positive eigenvalues of G and G' , respectively. If $\varrho \leq \varrho'$, then

$$(6.11) \quad \mu_k \leq \mu'_k \quad \text{for } k = 1, 2, 3, \dots$$

Proof. Since $\varrho \leq \varrho'$, we see that for any $u \in K^0(D)$ or $K^d(D)$,

$$\frac{\langle Gu, u \rangle}{\langle u, u \rangle} = \frac{\int_{\Gamma_e} \varrho \left(\frac{\partial u}{\partial \nu} \right)^2 ds}{\langle u, u \rangle^* + \int_{\Gamma_e} \varrho \left(\frac{\partial u}{\partial \nu} \right)^2 ds} \leq \frac{\int_{\Gamma_e} \varrho' \left(\frac{\partial u}{\partial \nu} \right)^2 ds}{\langle u, u \rangle^* + \int_{\Gamma_e} \varrho' \left(\frac{\partial u}{\partial \nu} \right)^2 ds} = \frac{\langle G'u, u \rangle'}{\langle u, u \rangle'},$$

which implies (6.11). \square

Proof of Theorem 1.1. a) First, let (\mathcal{M}, g) be a real analytic Riemannian manifold, and let the boundary $\partial\Omega$ of Ω be real analytic. We divide the domain $\bar{\Omega}$ into subdomains in the following manner. It is clear that the boundary $\partial\Omega$ of the domain Ω is the union of a finite number of closed pieces $\bar{\Gamma}_1, \dots, \bar{\Gamma}_p$ (without common inner point on the surface). Let U be a coordinate neighborhood which contains $\bar{\Gamma}_j$, let $x_i = x_i(Q)$ and $a_i = a_i(\nu_Q)$ be the coordinates of a point Q in $\bar{\Gamma}_j$ and the interior Riemannian normal ν_Q at Q , respectively. We define the subdomain D_j and surface Γ_j^σ by

$$D_j = \{P \mid x(P) = x(Q) + \xi_n a(\nu_Q), \quad Q \in \Gamma_j, \quad 0 < \xi_n < \sigma\}$$

and

$$\Gamma_j^\sigma = \{P | x(P) = x(Q) + \sigma a(\nu_Q), Q \in \Gamma_j\},$$

where σ is a positive constant. The closure of D_j is

$$(6.12) \quad \bar{D}_j = \{P | x(P) = x(Q) + \xi_n a(\nu_Q), Q \in \bar{\Gamma}_j, 0 \leq \xi_n \leq \sigma\}.$$

By the assumption, each $\bar{\Gamma}_j$, which is contained in a coordinate neighborhood, can be represented by equations

$$(6.13) \quad x_i = \psi_i(\xi_1, \dots, \xi_{n-1})$$

with real analytic functions ψ_i , i.e., it is the image of the closure $\bar{\Upsilon}_j$ of an open domain Υ_j of \mathbb{R}^{n-1} . Hence, if σ is sufficiently small, the definitions have a sense and the formula

$$(6.14) \quad x(P) = x(Q) + \xi_n a(\nu_Q), \quad Q \in \bar{\Gamma}_j, \quad 0 \leq \xi_n \leq \sigma$$

defines a real analytic homeomorphism of a neighborhood of the image of \bar{D}_j in \mathbb{R}^n given by the coordinates x and a neighborhood U_j of the closed cylinder \bar{F}_j in \mathbb{R}^n defined by $\bar{F}_j = \{\xi | (\xi_1, \dots, \xi_{n-1}) \in \bar{\Upsilon}_j, 0 \leq \xi_n \leq \sigma\}$. Moreover, the domains $\bar{D}_1, \dots, \bar{D}_p$ have no common inner points and the remainder $D_0 = \Omega - \cup_{j=1}^p \bar{D}_j$ of Ω has a finite number of connected parts. Note that the boundary of \bar{D}_0 contains no part of $\partial\Omega$.

Let us define the spaces $K = K(\Omega)$, \mathcal{K} and the transformation G as in Section 5. We shall investigate the asymptotic behavior of $A(\tau)$ with regard to transformation G on space \mathcal{K} . Moreover, we define the function spaces

$$\begin{aligned} K_j^0 &= \{u_j | u_j \in H_0^1(D_j) \cap H^2(D_j) \cap C^\infty(\bar{D}_j), \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^\sigma, \text{ and} \\ &\quad \Delta u = 0 \text{ on } \partial D_j - (\Gamma_j \cup \Gamma_j^\sigma)\}, \\ H_0^0 &= \{u_0 | u_0 \in H_0^1(D_0) \cap H^2(D_0), \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial D_0\}, \\ K_j^d &= \{u_j | u_j \in H^2(D_j), u_j = 0 \text{ on } \Gamma_j, u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } \Gamma_j^\sigma\}, \\ &\quad (j = 0, 1, \dots, p), \end{aligned}$$

and the bilinear functionals

$$(6.15) \quad \langle u_j, u_j \rangle_j^* = \int_{D_j} |\Delta_g u_j|^2 dR, \quad (j = 0, \dots, p),$$

$$(6.16) \quad [u_j, v_j]_j = \int_{\Gamma_j} \varrho \frac{\partial u_j}{\partial \nu} \frac{\partial v_j}{\partial \nu} ds, \quad (j = 1, \dots, p), \quad [u_0, v_0] = 0,$$

and

$$(6.17) \quad \langle u_j, v_j \rangle_j = \langle u_j, v_j \rangle_j^* + [u_j, v_j]_j, \quad (j = 0, \dots, p),$$

where $u_j, v_j \in K_j^0$ or K_j^d . Closing K_j^0 and K_j^d with respect to the norm $|u_j|_j = \sqrt{\langle u_j, u_j \rangle_j}$, we get the Hilbert spaces \mathcal{K}_j^0 and \mathcal{K}_j^d , ($j = 0, \dots, p$). In the same manner as in Section 5 we can define the Hilbert spaces \mathcal{K}^0 and \mathcal{K}^d , and then define the positive, completely continuous transformations G^0, G^d, G_j^0 and G_j^d on $\mathcal{K}^0, \mathcal{K}^d, \mathcal{K}_j^0$ and \mathcal{K}_j^d , respectively. Consequently, we can prove

$$(6.18) \quad A^0(\tau) \leq A(\tau) \leq A^d(\tau) \quad \text{for all } \tau,$$

and

$$(6.19) \quad A^0(\tau) = \sum_{j=0}^p A_j^0(\tau), \quad A^d(\tau) = \sum_{j=0}^p A_j^d(\tau),$$

where $A^0(\tau)$, $A^d(\tau)$, $A_j^0(\tau)$ and $A_j^d(\tau)$ are the numbers of eigenvalues of the transformations G^0 , G^d , G_j^0 and G_j^d on \mathcal{K}^0 , \mathcal{K}^d , \mathcal{K}_j^0 and \mathcal{K}_j^d which are greater than or equal to $(1 + \tau)^{-1}$, respectively.

Since $[u_0, u_0]_0 = 0$ for all $u_0 \in K_0^0$ or K_0^d and $\langle G_0^0 u_0, u_0 \rangle_0 = \langle G_0^d u_0, u_0 \rangle_0 = [u_0, u_0]_0$, we immediately find that $G_0^0 = G_0^d = 0$, so that $A_0^0(\tau) = A_0^d(\tau) = 0$, ($\tau \geq 0$). Thus we need estimate $A_j^0(\tau)$ and $A_j^d(\tau)$ for those domains D_j , where $\int_{\Gamma_j} \varrho ds > 0$.

We can choose a finer subdivision of $\partial\Omega$ by subdividing the domains $\bar{\Upsilon}_j$ into smaller ones, e.g. by means of a cubical net in the coordinates ξ . According to p.71 of [39], by performing a linear transformation Φ of the coordinates we can choose a new coordinate system (η) such that

$$g^{il}(\bar{\eta}) = \delta^{il}, \quad (i, l = 1, \dots, n),$$

for one point $\bar{\eta} \in T_j$, where $T_j := \Phi(\Upsilon_j)$. Setting $\phi_i = \psi_i \circ \Phi^{-1}$ and $\tilde{a}_i = a_i \circ \Phi^{-1}$, we see that

$$(6.20) \quad \begin{aligned} x_i(P) &= \phi_i(\eta_1, \dots, \eta_{n-1}) + \eta_n \tilde{a}_i(\nu(\eta_1, \dots, \eta_{n-1})), \\ \text{for } (\eta_1, \dots, \eta_{n-1}) &\in \bar{T}_j, \quad 0 \leq \eta_n \leq \sigma \end{aligned}$$

defines a real analytic homeomorphism from \bar{E}_j to the image of \bar{D}_j , where $\bar{E}_j = \{\eta = (\eta_1, \dots, \eta_n) | (\eta_1, \dots, \eta_{n-1}) \in \bar{T}_j, 0 \leq \eta_n \leq \sigma\}$ is a cylinder in \mathbb{R}^n (This can also be realized by choosing a (Riemannian) normal coordinates system at the point $\bar{\eta} \in T_j$ for the manifold (\mathcal{M}, g) (see, for example, p.77 of [23]) such that $a(\nu(\eta)) = (0, \dots, 0, 1)$ and by using the mapping (6.20).) If we denote the new subdomains of $\partial\Omega$ by $\bar{\Gamma}_j$ as before, it is clear that we can always choose them and σ (i.e., by letting σ sufficiently small and further making a finer subdivision of $\partial\Omega$, see p.71 of [39]), so that,

$$(6.21) \quad |g'^{il}(\eta') - g^{il}(\bar{\eta})| < \epsilon, \quad i, l = 1, \dots, n,$$

$$(6.22) \quad \left| \frac{1}{\sqrt{|g(\eta')|}} \frac{\partial}{\partial x_i} (\sqrt{|g(\eta')|} g^{il}(\eta')) - \frac{1}{\sqrt{|g(\bar{\eta})|}} \frac{\partial}{\partial x_i} (\sqrt{|g(\bar{\eta})|} g^{il}(\bar{\eta})) \right| < \epsilon, \\ i, l = 1, \dots, n,$$

for any given $\epsilon > 0$, and all points $\eta' \in \bar{E}_j$. The inequalities (6.21) imply that

$$(6.23) \quad (1 + \tilde{M}_j \epsilon)^{-1} \sum_{i=1}^n t_i^2 \leq \sum_{i,l=1}^n g^{il}(\eta') t_i t_l \leq (1 + \tilde{M}_j \epsilon) \sum_{i=1}^n t_i^2$$

for all points $\eta' \in \bar{E}_j$ and all real numbers t_1, \dots, t_n , where \tilde{M}_j is a positive constant depending only on g^{il} and \bar{E}_j (cf. Lemma 6.1). This and formula (128) of [39] say that

$$(6.24) \quad (1 + \epsilon \tilde{M}_j)^{-n/2} |T_j| \leq |\Gamma_j| \leq (1 + \epsilon \tilde{M}_j)^{n/2} |T_j|,$$

where

$$|\Gamma_j| = \int_{T_j} \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad |T_j| = \int_{T_j} d\eta \cdots d\eta_{n-1}$$

are the Riemannian and Euclidean areas of Γ_j and $|T_j|$, respectively,

Next, we consider the Hilbert spaces \mathcal{K}_j^0 and \mathcal{K}_j^d . When transported to \bar{E}_j , the underlying incomplete function spaces K_j^0 and K_j^d are

$$K_j^0 = \{u | u \in H_0^1(E_j) \cap H^2(E_j) \cap C^\infty(\bar{E}_j), \frac{\partial u}{\partial \nu} = 0 \text{ on } T_j^\sigma, \text{ and } \Delta u = 0 \text{ on } \partial E_j - (T_j \cup T_j^\sigma)\}$$

and

$$K_j^d = \{u_j | u_j \in H^2(E_j), u_j = 0 \text{ on } T_j, u_j = \frac{\partial u_j}{\partial \nu} = 0 \text{ on } T_j^\sigma\},$$

respectively. The inner product, which is similar to Section 5, is defined by

$$\langle u, v \rangle_j = \int_{E_j} (\Delta_g u)(\Delta_g v) \sqrt{g(\eta)} d\eta_1 \cdots d\eta_n + \int_{T_j} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}$$

and the transformations G_j^0 and G_j^d are defined by

$$\langle G_j^0 u, v \rangle_j = \int_{T_j} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u, v \text{ in } \mathcal{K}_j^0,$$

and

$$\langle G_j^d u, v \rangle_j = \int_{T_j} \varrho \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \sqrt{g(\eta)} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u, v \text{ in } \mathcal{K}_j^d,$$

respectively.

Put

$$(6.25) \quad \underline{\varrho}_j = \inf_{\Gamma_j} \varrho \quad \text{and} \quad \bar{\varrho}_j = \sup_{\bar{\Gamma}_j} \varrho,$$

and let us introduce the inner products

$$\underline{\langle u, v \rangle}_j = \int_{E_j} (\Delta u)(\Delta v) d\eta_1 \cdots d\eta_n + \int_{T_j} \underline{\varrho}_j \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\eta_1 \cdots d\eta_{n-1}$$

and

$$\overline{\langle u, v \rangle}_j = \int_{E_j} (\Delta u)(\Delta v) d\eta_1 \cdots d\eta_n + \int_{T_j} \bar{\varrho}_j \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\eta_1 \cdots d\eta_{n-1}$$

in the spaces K_j^0 and K_j^d , respectively. By closing these spaces in the corresponding norms, we get Hilbert spaces $\underline{\mathcal{K}}_j^0$ and $\bar{\mathcal{K}}_j^d$. Furthermore, we obtain the positive, completely continuous transformations \underline{G}_j^0 and \bar{G}_j^d on $\underline{\mathcal{K}}_j^0$ and $\bar{\mathcal{K}}_j^d$, which are given by

$$(6.26) \quad \underline{\langle \underline{G}_j^0 u, v \rangle}_j = \int_{T_j} \underline{\varrho}_j \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u \text{ and } v \text{ in } \underline{\mathcal{K}}_j^0$$

and

$$(6.27) \quad \overline{\langle \bar{G}_j^d u, v \rangle}_j = \int_{T_j} \bar{\varrho}_j \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\eta_1 \cdots d\eta_{n-1}, \quad \text{for } u \text{ and } v \text{ in } \bar{\mathcal{K}}_j^d,$$

respectively.

Let $\mu_k(G_j^0)$ be the k -th positive eigenvalue of G_j^0 and so on. According to Lemma 6.1 and Remark 6.2, $\Lambda_1^d(D_j)$ and $\Theta_1^d(D_j)$ have uniformly positive lower bound when repeated taking finer division of D (In fact, by repeated halving the side length of every rectangular parallelepiped in the partition net of the coordinates η for each cylinder E_j , we see that $\Lambda_1^d(D_j)$ will tend to $+\infty$, and that $\Theta_1^d(D_j)$ will have a positive lower bound). This implies that the corresponding positive constants \tilde{M}_j and M_j have uniformly upper bound when we further divide the domain D into finer a division, where \tilde{M}_j is defined as before, and M_j is a constant independent of ϵ and depending only on g , $\frac{\partial g_{im}}{\partial x_l}$, $\frac{\partial g^{im}}{\partial x_l}$ and \bar{E}_j as in Lemma 6.1. Denote by $c_j(\epsilon)$ the maximum value of $(1 + \epsilon \tilde{M}_j)^{(n+1)/2} (\max\{(1 + \epsilon M_j), (1 + \epsilon \tilde{M}_j)^{(n+1)/2}\})$

and $(1 + \epsilon \tilde{M}_j)^{(n+1)/2} \left(\min\{(1 - \epsilon M_j), (1 + \epsilon \tilde{M}_j)^{-(n+1)/2}\} \right)^{-1}$. Obviously, $c_j(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. By virtue of (6.21) and (6.22), it follows from Lemmas 6.1 and 6.3 that

$$(6.28) \quad \mu_k(G_j^d) \leq c_j(\epsilon) \mu_k(\bar{G}_j^d)$$

and

$$(6.29) \quad \mu_k(G_j^0) \geq c_j(\epsilon)^{-1} \mu_k(\underline{G}_j^0),$$

so that

$$A_j^d(\tau) \leq \bar{A}_j^d(c_j(\epsilon)\tau + c_j(\epsilon) - 1)$$

and

$$(6.30) \quad A_j^0(\tau) \geq \underline{A}_j^0(c_j(\epsilon)^{-1}\tau + c_j(\epsilon)^{-1} - 1)$$

where $\bar{A}_j^d(\tau)$ and $\underline{A}_j^0(\tau)$ are the numbers of eigenvalues of the transformation $\bar{G}_j^d(1 + \tau)^{-1}$, respectively. and \underline{G}_j^0 which are greater than or equal to By (6.18) and (6.19), we obtain

$$(6.31) \quad \begin{aligned} \sum_j \underline{A}_j^0(c_j(\epsilon)^{-1}\tau + c_j(\epsilon)^{-1} - 1) &\leq A(\tau) \\ &\leq \sum_j \bar{A}_j^d(c_j(\epsilon)\tau + c_j(\epsilon) - 1). \end{aligned}$$

Finally, we shall apply the results of Section 5 to estimate $\underline{A}_j^0(\tau)$ and $\bar{A}_j^d(\tau)$. Note that

$$(6.32) \quad \bar{A}_j^d(\tau) \leq \bar{A}_j^f(\tau) \quad \text{for all } \tau > 0,$$

where \bar{A}_j^f is defined similarly to (5.15)–(5.17). It follows from (5.49), (5.66), (6.32) and (5.69)–(5.71) that

$$(6.33) \quad \lim_{\tau \rightarrow +\infty} \frac{\underline{A}_j^0(\tau)}{\tau^{n-1}} \geq \omega_{n-1}(4\pi)^{-(n-1)} |T_j| \underline{\varrho}_j^{n-1}$$

and

$$(6.34) \quad \lim_{\tau \rightarrow +\infty} \frac{\bar{A}_j^f(\tau)}{\tau^{n-1}} \leq \omega_{n-1}(4\pi)^{-(n-1)} |T_j| \bar{\varrho}_j^{n-1},$$

where $|T_j|$ is the Euclidean area of T_j . By (6.31), (6.32), (6.33), (6.34), (6.18), (6.19) and (6.24), we find that

$$\lim_{\tau \rightarrow \infty} A(\tau) \tau^{-(n-1)} \leq \omega_{n-1}(4\pi)^{-(n-1)} \tilde{c}_j(\epsilon) \sum_j \bar{\varrho}_j^{n-1} |\Gamma_j|,$$

and

$$\lim_{\tau \rightarrow \infty} A(\tau) \tau^{-(n-1)} \geq \omega_{n-1}(4\pi)^{-(n-1)} \tilde{c}_j(\epsilon)^{-1} \sum_j \underline{\varrho}_j^{n-1} |\Gamma_j|,$$

where $\tilde{c}_j(\epsilon) = (1 + \epsilon \tilde{M}_j)^{n/2} c_j(\epsilon)^{n-1}$. Note that ϱ is Riemannian integrable since it is non-negative bounded measurable function on Γ_ϱ . Therefore, letting $\epsilon \rightarrow 0$, we obtain the desired result that

$$(6.35) \quad A(\tau) \sim \frac{\omega_{n-1} \tau^{n-1}}{(4\pi)^{(n-1)}} \int_{\partial\Omega} \varrho^{n-1} ds \quad \text{as } \tau \rightarrow +\infty.$$

b) Next, since a C^2 -smooth metric g can be approximated in C^2 by a metric g'_ϵ which is C^2 -smooth on \mathcal{M} and piecewise real analytic (i.e., g'_ϵ is C^2 -smooth and g'_ϵ is composed

of a finite number of real analytic functions) in any compact submanifold of (\mathcal{M}, g) such that

$$|g_\epsilon'^{il} - g^{il}| < \epsilon, \quad i, l = 1, \dots, n,$$

$$\left| \frac{1}{\sqrt{|g_\epsilon'|}} \frac{\partial}{\partial x_i} (\sqrt{|g_\epsilon'|} g_\epsilon'^{il}) - \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} (\sqrt{|g|} g^{il}) \right| < \epsilon, \quad i, l = 1, \dots, n,$$

for all points in \bar{D} , with any given $\epsilon > 0$. In addition, any bounded domain D with C^2 -smooth boundary can also be approximated (see, the definition in Section 2) by domain D'_ϵ with C^2 -smooth and piecewise real analytic boundary. Thus, the methods of Lemma 6.1 and a) still work in this case, so that we can estimate the eigenvalues for $g_\epsilon'^{il}$ in D'_ϵ . But for these eigenvalues (6.35) is true. Therefore, letting $\epsilon \rightarrow 0$ and noticing that $ds'_\epsilon \rightarrow ds$, we get that (6.35) also holds for the C^2 -smooth metric g^{il} and D . \square

c) With the same arguments as in the case b), we immediately see that the formula (1.8) is still true for a bounded domain with a piecewise C^2 -smooth boundary in a C^1 -smooth Riemannian manifold.

Remark 6.4. Our method in the proof of Theorem 1.1 is new and significantly different from that of [39]. In [39] Sandgren used a technique of Lipschitz image of a convex subset for the harmonic Steklov problem. In our proof, Γ_j needn't be the image of a convex subset. Next, in order to estimate $A^d(\tau)$, we introduce a new counting function $A^f(\tau)$ as done in Section 5. In addition, we use the uniform boundedness of the constants M_j and \bar{M}_j to estimate the asymptotic behavior for any finer division according to Lemma 6.1.

Proof of Corollary 1.2. By (1.8), we have

$$(6.36) \quad A(\lambda_k) \sim \frac{\omega_{n-1} \lambda_k^{n-1}}{(4\pi)^{(n-1)}} (\text{vol}(\partial\Omega)), \quad \text{as } k \rightarrow +\infty.$$

Since $A(\lambda_k) = k$, we obtain (1.9), which completes the proof. \square

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